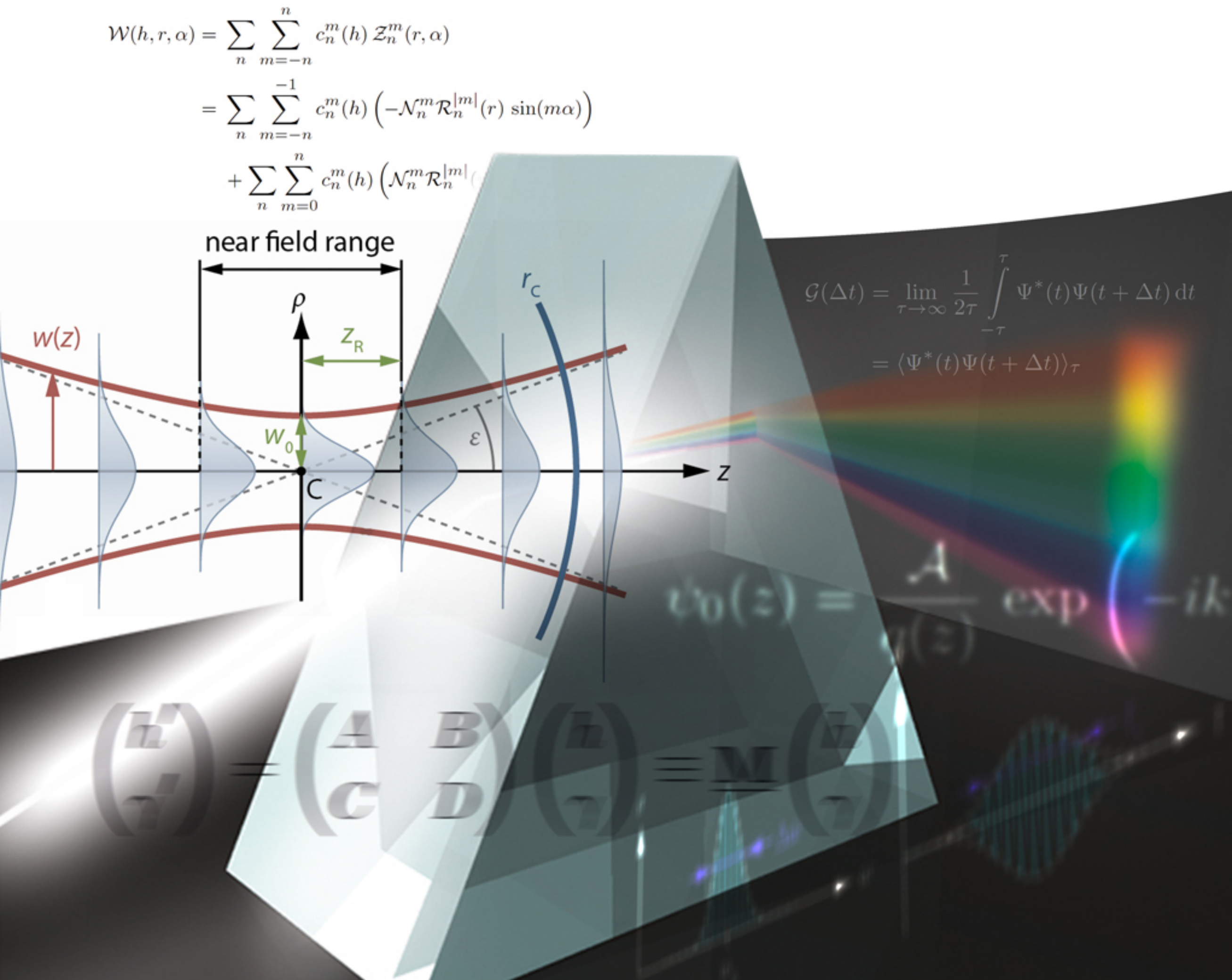
Appendix A

Michael Kaschke, Karl-Heinz Donnerhacke
and Michael Stefan Rill

Layout by Kerstin Willnauer

Basics of Optics



Solutions to Problems in „Optical Devices in Ophthalmology and Optometry“

Contents

PA.1
Optics on a summer day

It is a warm day and you are standing on the edge of a swimming pool. You are wearing Polaroid sunglasses which obviously reduce the glare from sunlight reflected from the water surface ($n_{\mathrm{water}} = 1.33$).

1. Which polarization direction is blocked by your sunglasses?
2. If the water surface is observed at a certain angle, the glare of the sunlight is perfectly filtered out by the sunglasses. Derive the conditional equation for this angle from the Fresnel equations (A63) – (A66).
3. When looking at the opposite side of the swimming pool, you realize that the bottom edge appears to be at an angle of $30°$ below the horizontal. But when you sit on the pool edge, the bottom edge of the opposite side of the pool appears to be at an angle of $14°$ below the horizontal. Determine length and depth of the pool. *Hint:* Estimate the height of your eyes above the surface of the water when standing and sitting.
4. You now gaze directly into the sun. Estimate the time it takes for a photon to travel from the surface of the sun to your retina. What is the additional time delay if you are wearing sunglasses?
5. You replace your Polaroid sunglasses by a special type of filter glasses. Only light with a wavelength of $550\,\mathrm{nm}$ can pass through these glasses. Calculate the number of photons that enter your eye if you look for 0.1 s at the sun. What energy is absorbed by your eye during that time (all photons shall be absorbed). *Hints:* You need to use some results of Appendix B; power output of the sun: $P_{\mathrm{sun}} = 4.2\times10^{26}$ W; distance between Earth and sun: $d_{\mathrm{ES}} = 1.5\times10^{11}$ m; pupil diameter: $d_{\mathrm{p}} = 1.5\,\mathrm{mm}$.
6. In order to cool down, you are jumping into the pool. As discussed in Chapter 1, human eyes perceive color via three types of cones. Explain why the color of an object that appears blue in air also appears blue underwater although the speed of light (and hence its wavelength) is shortened.

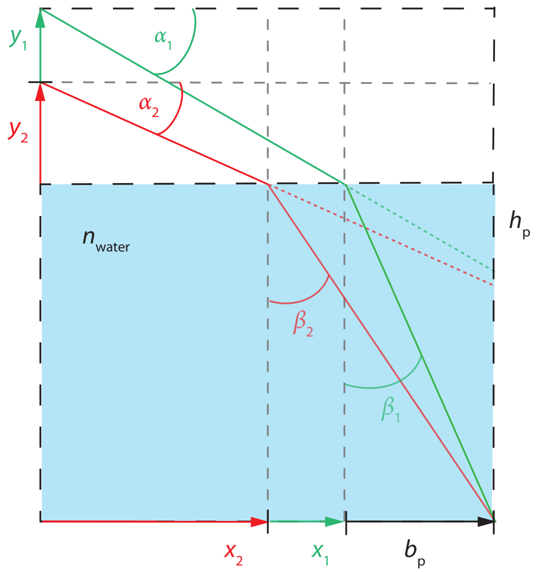


Figure SA.1 Geometrical and optical description of a person standing nearby a swimming pool.

Solution:

1. It is the parallel polarization of the reflected light, that is, light for which the polarization is perpendicular to the plane of incidence (defined by the propagation vector and the normal to the refracting surface) (Figure A.30).

2. See Problem PA.9.

3. The setup to be discussed is illustrated in Figure SA.1. We estimate the height of your eyes above the water surface when standing and sitting to be 180 cm and 90 cm, respectively. Thus, the corresponding angles read

$$\alpha_1 = 30^\circ \quad ,$$
$$\alpha_2 = 14^\circ \quad .$$

From Figure SA.1 we additionally deduce

$$\sin\beta_1 = \frac{\sin\alpha_1}{n_{\mathrm{water}}} \quad ,$$
$$\sin\beta_2 = \frac{\sin\alpha_2}{n_{\mathrm{water}}} \quad ,$$
$$x_1 = \frac{y_1}{\tan\alpha_1} \quad ,$$
$$x_2 = \frac{y_2}{\tan\alpha_2} \quad ,$$
$$b_{\mathrm{p}} - x_1 = h_{\mathrm{p}}\tan\beta_1 \quad ,$$
$$b_{\mathrm{p}} - x_2 = h_{\mathrm{p}}\tan\beta_2 \quad .$$

From these relations, we can directly calculate

$$b_{\mathrm{p}} = \frac{x_1 - \left(\frac{\tan\beta_1}{\tan\beta_2}\right) x_2}{1 - \frac{\tan\beta_1}{\tan\beta_2}} = 4.01 \text{ m}$$

and

$$h_{\mathrm{p}} = \frac{b_{\mathrm{p}} - x_1}{\sin\beta_1} = 2.33 \text{ m} \quad .$$

4. It takes 8 minutes for a photon to travel 150×10^6 km from the sun to the Earth. The additional delay by the sun glasses can be calculated via $(n-1)d/c$, where n is the refractive index of the glass, d its thickness, and c the speed of light. With reasonable values, we obtain a delay of 5 ps.

5. The power incident on the pupil of the eye is given by

$$P_{\mathrm{pupil}} = \frac{A_{\mathrm{pupil}}}{A_{\mathrm{sphere}}} \cdot P_{\mathrm{sun}}$$

with the surface of a sphere at the Earth's distance from the sun $A_{\mathrm{sphere}} = 4\pi d_{\mathrm{ES}}^2$ and the cross-section of the eye pupil $A_{\mathrm{pupil}} = \pi d_{\mathrm{p}}^2/4$. The rate at which the photons enter the eye through the pupil is given by

$$\frac{\Delta N}{\Delta t} = \frac{P_{\mathrm{pupil}}}{E_{\mathrm{photon}}}$$

with $E_{\mathrm{photon}} = hc/\lambda$ being the energy of a single photon. As a consequence, we obtain

$$\frac{\Delta N}{\Delta t} = \frac{P_{\mathrm{sun}}\lambda}{16hc} \cdot \left(\frac{d_{\mathrm{p}}}{d_{\mathrm{ES}}}\right)^2 \approx 10^{16} \quad .$$

With the numerical values, we finally calculate with $\Delta t = 0.1$ s

$$\Delta N = 7.28 \times 10^{14} \approx 10^{15} \quad .$$

The corresponding energy absorbed by the eye during that time is

$$E = \Delta N \cdot E_{\mathrm{photon}} \approx 0.3 \text{ mJ} \quad .$$

6. The photoreceptors are excited by photons of a certain energy. The energy of a photon is given by

$$E_{\text{photon}} = h\nu = h\frac{c_0}{\lambda_0} = h\frac{c_{\text{n}}}{\lambda_{\text{n}}} = h\frac{c_0/n}{\lambda_0/n} \quad .$$

As the photon frequency remains constant, the photon energy does not change when it travels through a medium with refractive index n, although the wavelength changes.

PA.2
Atmospheric refraction

Imagine you stand at the ocean shore to watch the setting sun. Interestingly, you can already observe the first rays, although the sun is actually at an angle α below the horizon.

1. Calculate the angle $\alpha(r, n, h)$. Assume that the Earth's atmosphere has a uniform refractive index ($n = 1.0003$) and extends to a height of $h = 20\,\text{km}$. Beyond the atmosphere, there is vacuum. The Earth's radius is $r = 6378\,\text{km}$.
2. Why does the sun appear to be flattened during the setting?

Solution:

1. As shown in Figure SA.2, the rays from the sun are incident at an angle α below the horizon. The rays are refracted at point B – assuming the atmosphere to be a shell of height h – and reach the observer at point A. n is the refractive index of Earth's atmosphere and n_0 that of vacuum. Applying Snell's law at point B leads to

$$\begin{aligned} n_0 \sin(\theta + \alpha) &= n \sin\theta \quad , \\ n_0 \sin\theta \cos\alpha + n_0 \cos\theta \sin\alpha &= n \sin\theta \quad , \\ n_0 \cos\theta \sin\alpha &= \sin\theta (n - n_0 \cos\alpha) \quad , \\ \frac{n_0}{\tan\theta} &= \frac{n}{\sin\alpha} - \frac{n_0}{\tan\alpha} \quad . \end{aligned}$$

With the approximation $\sin\alpha \approx \tan\alpha \approx \alpha$, we obtain

$$\alpha = \left(\frac{n - n_0}{n_0}\right) \tan\theta \quad .$$

For the triangle OAB, we use Pythagoras' theorem and write

$$\begin{aligned} \overline{\text{AB}}^2 &= (r + h)^2 - r^2 = (2r + h)h \quad , \\ \tan\theta &= \frac{r}{\overline{\text{AB}}} = \frac{r}{\sqrt{h(2r + h)}} \quad , \\ \alpha &= \left(\frac{n - n_0}{n_0}\right) \frac{r}{\sqrt{h(2r + h)}} \quad . \end{aligned}$$

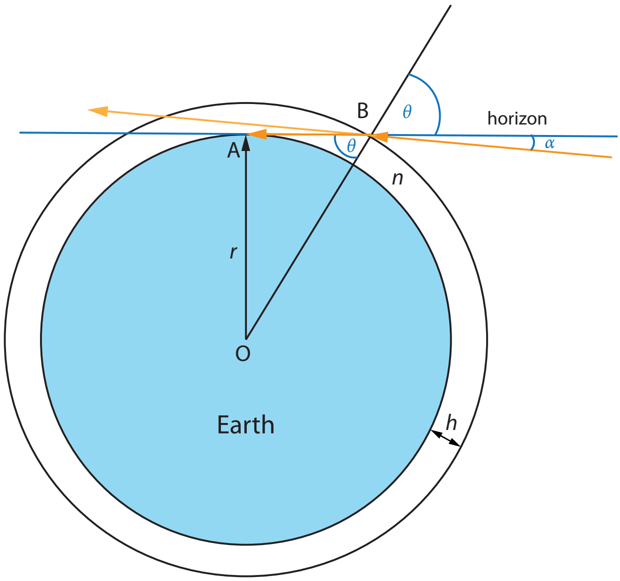


Figure SA.2 Ray diagram to describe atmospheric refraction.

Using the values $n = 1.0003$, $n_0 = 1$, $h = 20\,\text{km}$, $r = 6378\,\text{km}$, we find

$$\alpha = \left(\frac{1.0003 - 1}{1}\right) \frac{6378}{\sqrt{20(2 \cdot 6378 + 20)}} = 0.217^\circ \quad .$$

2. During sunset, the part of the sun which is exactly incident at an angle α below the horizon is imaged on the horizon being almost perfectly flat due to the very large radius of curvature of 6378 km. Thus, the sun appears to be flat during sunset.

PA.3
Lens maker's equation

Derive the lens maker's equation for a thin and a thick lens.

Solution:

Figure SA.3 shows a thick lens with thickness L. The lens with a refractive index of n_2 is embedded in a medium with a refractive index of n_1. The change in ray height at the two surfaces cannot be ignored due to the thickness of the lens. Surface 1 and 2 have a curvature of

$$K_1 = \frac{1}{r_1}, \quad K_2 = -\frac{1}{r_2} \quad ,$$

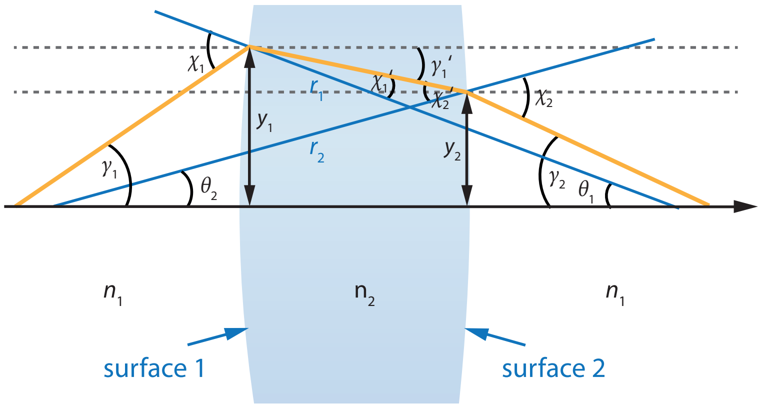


Figure SA.3 Light passing through a thick lens.

where r_1 and r_2 are the radii of curvature, respectively. At surface 1, we have

$$\chi_1 = y_1 K_1 \quad ,$$
$$n_1 \chi_1 = n_2 \chi_1'$$
$$\Rightarrow \frac{n_1}{n_2}\chi_1 = \chi_1' \quad ,$$
$$\gamma_1' = \chi_1 - \chi_1' = y_1 K_1 - \frac{n_1}{n_2} y_1 K_1 \quad .$$

In paraxial approximation, the relationship between the two ray heights is

$$y_2 = y_1 - L\gamma_1' \quad ,$$
$$\gamma_1' = \chi_1 - \chi_1' = y_1 K_1 - \frac{n_1}{n_2} y_1 K_1 \quad ,$$
$$y_2 = y_1 - L y_1 K_1 + \frac{n_1}{n_2} L y_1 K_1 \quad . \tag{SA.1}$$

At surface 2, we have

$$n_2 \chi_2' = n_1 \chi_2 \quad ,$$
$$\chi_2' = y_2 K_2 + \gamma_1' \quad ,$$
$$\chi_2 = y_2 K_2 + \gamma_2 \quad ,$$
$$n_2 y_2 K_2 + n_2 \gamma_1' = n_1 y_2 K_2 + n_1 \gamma_2 \quad ,$$
$$\gamma_1' = \frac{y_1 - y_2}{L} \quad ,$$
$$\Rightarrow n_2 y_2 K_2 + \frac{n_2 y_1}{L} - \frac{n_2 y_2}{L} = n_1 y_2 K_2 + n_1 \gamma_2 \quad . \tag{SA.2}$$

Substituting y_2 from Eq. (SA.1) into Eq. (SA.2) leads to

$$n_2 y_2 K_2 + n_2 y_1 K_1 - n_1 y_1 K_1 = n_1 y_2 K_2 + n_1 \gamma_2 \quad ,$$
$$(n_2 - n_1) y_2 K_2 + (n_2 - n_1) y_1 K_1 = n_1 \gamma_2 \quad .$$

Dividing both sides by y_1 yields

$$\frac{(n_2 - n_1)y_2K_2}{y_1} + (n_2 - n_1)K_1 = \frac{n_1\gamma_2}{y_1} \ . \tag{SA.3}$$

Using Eq. (SA.1) and substituting it into Eq. (SA.3) then yields

$$\begin{aligned}
\frac{y_2}{y_1} &= \left(1 - \frac{(n_2 - n_1)LK_1}{n_2}\right) \ , \\
(n_2 - n_1)\left(1 - \frac{(n_2 - n_1)LK_1}{n_2}\right)K_2 + (n_2 - n_1)K_1 &= \frac{n_1\gamma_2}{y_1} \ , \\
(n_2 - n_1)K_2 - \frac{(n_2 - n_1)^2LK_1K_2}{n_2} + (n_2 - n_1)K_1 &= \frac{n_1\gamma_2}{y_1} \ , \\
\frac{(n_2 - n_1)}{n_1}\left(K_1 + K_2 - \frac{(n_2 - n_1)LK_1K_2}{n_2}\right) &= \frac{\gamma_2}{y_1} \ .
\end{aligned}$$

The focal length of the lens is measured from the principal plane and equals

$$\frac{y_1}{f} = \gamma_2 \ .$$

Finally, substituting for K_1, K_2 and f leads to

$$\frac{(n_2 - n_1)}{n_1}\left(\frac{1}{r_1} - \frac{1}{r_2} + \frac{(n_2 - n_1)L}{n_2r_1r_2}\right) = \frac{1}{f'} \ .$$

We know from Eq. (A14) that

$$\begin{aligned}
&\frac{1}{s'} - \frac{1}{s} = \frac{1}{f'} \\
\Rightarrow &\frac{(n_2 - n_1)}{n_1}\left(\frac{1}{r_1} - \frac{1}{r_2} + \frac{(n_2 - n_1)L}{n_2r_1r_2}\right) = \frac{1}{s'} - \frac{1}{s} \ .
\end{aligned} \tag{SA.4}$$

Equation (SA.4) is called the lens maker's equation for a thick lens. If $L = 0$, the equation reduces to the lens maker's equation for a thin lens given by

$$\frac{(n_2 - n_1)}{n_1}\left(\frac{1}{r_1} - \frac{1}{r_2}\right) = \frac{1}{s'} - \frac{1}{s} \ .$$

An alternative approach to derive the lens maker's equation is by using the ABCD matrix method according to Problem PA.4.

PA.4
Galilei's telescope

In 1610, Galileo Galilei discovered the four moons of Jupiter using an afocal telescope. Although such an optical system does not alter the divergence of an incident bundle of rays, it *does* alter the width of the beam and thus increases the magnification. Galilei's device consisted of a thin, negative lens (ocular lens with focal length: $f_1 = -5\,\text{cm}$) and a thin, positive lens (objective lens with focal length: $f_2 = 80\,\text{cm}$).

1. Derive the imaging equation of a thin lens (lens maker's equation) by applying Eq. (A11) twice and by using the ABCD matrix of a spherical surface in Table A.1.
2. Use the ABCD matrix approach to derive the imaging equation for a thick lens (A16).
3. Derive the ABCD matrix for the Galilei telescope. What is the condition for an afocal optical system? Was the observed image of Jupiter's moons upright or inverted?
4. Just one year later, Johannes Kepler showed that telescopes can also be made of two positive lenses. The lenses were separated by the sum of their focal lengths. Check, by using the ABCD matrix approach, whether the observed image was upright or inverted?
5. We now use Galilei's telescope as a laser beam expander. For this purpose, we consider a Nd:YAG laser ($\lambda = 1064\,\text{nm}$) which emits a Gaussian beam with a waist radius of $w_0 = 1.3\,\text{mm}$. Calculate the resulting beam diameter after passage through the afocal Galilei telescope.
6. The expanded laser beam shall be focused with another lens so that the peak intensity does not fall below 80% within a distance of $\Delta z = 1\,\text{mm}$. What is the minimum focal length of this lens? How large is the beam diameter at the focal point?

Solution:

1. In Example A.2 (Section A.1.3), the approach is given for a thin lens.

2. Here, we show the same approach for a thick lens. We start from Eq. (A19) with

$$\begin{pmatrix} h' \\ \gamma' \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} h \\ \gamma \end{pmatrix} = \underline{\mathbf{M}} \begin{pmatrix} h \\ \gamma \end{pmatrix} \quad . \tag{SA.5}$$

   The coefficients of the ABCD matrix can be generalized to

   - A: Lateral magnification for $\gamma = 0$
   - D: Angular magnification
   - C: Refractive power for $\gamma = 0$

A thick lens is a combination of two spherical surfaces with a finite distance L in between. The ABCD matrix can thus be written as

$$\underline{\mathbf{M}} = \underbrace{\begin{pmatrix} 1 & 0 \\ \frac{n'-n}{nr_2} & \frac{n'}{n} \end{pmatrix}}_{\text{right surface}} \underbrace{\begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix}}_{\text{thickness}} \underbrace{\begin{pmatrix} 1 & 0 \\ \frac{n-n'}{n'r_1} & \frac{n}{n'} \end{pmatrix}}_{\text{left surface}} ,$$

$$\underline{\mathbf{M}} = \underbrace{\begin{pmatrix} 1 & 0 \\ \frac{n'-n}{nr_2} & \frac{n'}{n} \end{pmatrix}}_{\text{right surface}} \begin{pmatrix} 1 + \frac{(n-n')L}{n'r_1} & \frac{Ln}{n'} \\ \frac{n-n'}{n'r_1} & \frac{n}{n'} \end{pmatrix} ,$$

$$\underline{\mathbf{M}} = \begin{pmatrix} 1 - \frac{(n'-n)L}{n'r_1} & \frac{Ln}{n'} \\ \frac{n'-n}{nr_2}\left(1 - \frac{L(n'-n)}{n'r_1}\right) - \frac{n'-n}{nr_1} & 1 + L\left(\frac{n'-n}{n'r_2}\right) \end{pmatrix} . \tag{SA.6}$$

Therefore, the refractive power is

$$\frac{1}{f} = C = \frac{n'-n}{nr_2}\left(1 - \frac{L(n'-n)}{n'r_1}\right) - \frac{n'-n}{n'r_1} . \tag{SA.7}$$

3. The ABCD matrix is given by

$$\underline{\mathbf{M}} = \underbrace{\begin{pmatrix} 1 & 0 \\ -\frac{1}{f_2} & 1 \end{pmatrix}}_{\text{positive lens}} \underbrace{\begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix}}_{\text{length}} \underbrace{\begin{pmatrix} 1 & 0 \\ -\frac{1}{f_1} & 1 \end{pmatrix}}_{\text{negative lens}} \tag{SA.8}$$

$$\Rightarrow \underline{\mathbf{M}} = \begin{pmatrix} 1 - \frac{z}{f_1} & z \\ \frac{z}{f_1 f_2} - \frac{1}{f_1} - \frac{1}{f_2} & 1 - \frac{z}{f_2} \end{pmatrix} . \tag{SA.9}$$

In an afocal optical setup, the image is formed at infinity. Hence, the refractive power is zero, that is, for an afocal system, the coefficient $C = 0$ and consequently

$$\frac{z}{f_1 f_2} - \frac{1}{f_1} - \frac{1}{f_2} = 0 \tag{SA.10}$$

$$\Rightarrow z = f_1 + f_2 . \tag{SA.11}$$

With Eqs. (SA.9) and (SA.11), the ABCD matrix now becomes

$$\underline{\mathbf{M}} = \begin{pmatrix} -\frac{f_2}{f_1} & f_1 + f_2 \\ 0 & -\frac{f_1}{f_2} \end{pmatrix} = \begin{pmatrix} 16 & 75 \\ 0 & 0.0625 \end{pmatrix} .$$

The lateral magnification is given by coefficient A so that

$$\beta = -\frac{f_2}{f_1} = \frac{80}{5} = 16\times .$$

Since the lateral magnification is positive, the observed image of the Jupiter moons was upright.

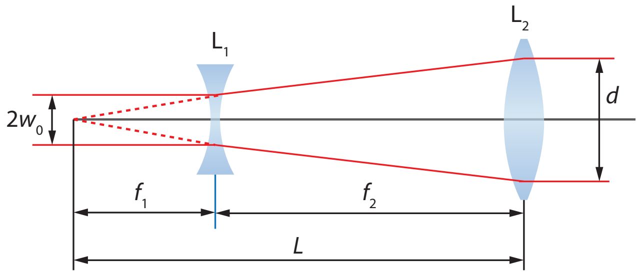


Figure SA.4 Galilei's telescope consisting of a negative and a positive lens.

4. The ABCD matrix for a telescope with two positive lenses is the same as for the Galilei telescope. However, due to two positive lenses, the focal lengths are both positive which renders the lateral magnification negative. As a consequence, the image observed would be inverted.

5. The principle setup is shown in Figure SA.4. The matrix of the system reads

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -\frac{1}{f_2} & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ -\frac{1}{f_1} & 1 \end{pmatrix}$$
$$= \begin{pmatrix} 1 - \frac{L}{f_1} & L \\ -\frac{1}{f_1} - \frac{1}{f_2} + \frac{L}{f_1 f_2} & 1 - \frac{L}{f_2} \end{pmatrix} \qquad \text{(SA.12)}$$

The system is afocal if for $\gamma = 0$, γ' is always zero for all ray heights h. From Eq. (SA.5), it follows that

$$\gamma' = C \cdot h + D \cdot \gamma \ .$$

Thus, we have an afocal system if $C = 0$. In our case of an afocal Galilei system, this is equivalent to

$$-\frac{1}{f_1} - \frac{1}{f_2} + \frac{L}{f_1 f_2} = 0$$
$$\Rightarrow L = f_1 + f_2 \ . \qquad \text{(SA.13)}$$

With Eqs. (SA.12) and (SA.13), the ABCD matrix for the afocal Galilei system then reads

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} -\frac{f_2}{f_1} & f_1 + f_2 \\ 0 & -\frac{f_1}{f_2} \end{pmatrix} \ .$$

The beam expansion A is the inverse of the angular magnification and thus given by

$$\frac{1}{\Gamma} = A = -\frac{f_2}{f_1} = \frac{d}{2w_0} = \frac{h_2}{h} = 16 \ .$$

The beam diameter then results to $d = 41.6\,\text{mm}$.

6. The evolution of the intensity of the Gaussian beam along the propagation direction is given by

$$I(z) = \frac{I_0}{1 - \left(\frac{z}{z_\mathrm{R}}\right)^2} \quad , \tag{SA.14}$$

in which z_R is the Rayleigh length. Here, we assumed that the waist is located at $z = 0$.
The required depth of focus of $\Delta z = 1\,\text{mm}$ leads to

$$I\left(z = \frac{\Delta z}{2}\right) = 0.8 \cdot I_0 \quad .$$

From this, we obtain

$$0.8 \cdot I_0 = \frac{I_0}{1 + \left(\frac{\Delta z/2}{z_\mathrm{R}}\right)^2}$$
$$\Rightarrow \frac{5}{4} = 1 + \left(\frac{\Delta z/2}{z_\mathrm{R}}\right)^2 \quad .$$

With $\Delta z = 1\,\text{mm}$, it follows that $z_\mathrm{R} = 1\,\text{mm}$. Using the definition of the Rayleigh length in Eq. (A83), the spot radius for a wavelength of $\lambda = 1064\,\text{nm}$ results to

$$w_0 = \sqrt{\frac{\lambda z_\mathrm{R}}{\pi}} = 0.0184\ \text{mm} \approx 18\ \mu\text{m} \quad .$$

Therefore, at the focal spot, the beam radius is 0.0184 mm. The minimum focal length can be derived from the equation for the divergence angle (A87) so that

$$\varepsilon = \frac{\lambda}{\pi w_0} = \frac{w_0}{z_\mathrm{R}} = 0.0184\ \text{rad} \quad .$$

The angle can also be calculated by using the beam diameter prior to focusing and the focal length of the focusing lens. Then, we have

$$\varepsilon = \frac{d}{2f}$$
$$\Rightarrow f = \frac{d}{2\varepsilon} = 1130\ \text{mm} \quad .$$

PA.5
LED–Fiber Coupling

Consider a GaAs LED ($n_{\text{GaAs}} = 3.4$) with a flat surface. The setup can be considered as a point source which is located close to the GaAs–air surface. At a distance of 2 mm, we place a silica fiber ($n_{\text{silica}} = 1.46$; core diameter $d_c = 1$ mm) which has a maximum acceptance angle of 14° in air.

1. What fraction of light (percentage) emitted by the active region of the LED can be coupled into the fiber? How does this value change if we fill the volume between fiber and GaAs LED with water? Neglect the reflection losses at both media interfaces.
2. By looking at the fiber, you realize that it has a cladding. For your setup, you need to know what kind of material has been used for the fiber. In a specification list, you found three possible options: $n_{c1} = 1.493$, $n_{c2} = 1.440$, or $n_{c3} = 1.430$. Which cladding material has been used in this case?
3. How can the LED fiber coupling be understood in the concept of the Helmholtz–Lagrange invariant?

Solution:

1. A point source at the GaAs–air interface can be considered as a Lambertian radiator. The latter is a light source with constant radiance L which does not depend on the observation direction (Figure SA.5). The radiant intensity (Table A.4) in a direction θ to the surface normal is

$$I(\theta) = L \cdot A \cdot \cos\theta = I_0 \cos\theta$$

with A being a spherical surface. By integrating over the hemisphere, we obtain the total power of the Lambertian radiator

$$P = \int I(\theta)\mathrm{d}\Omega = I_0 \int_0^{\pi/2} 2\pi \sin\theta \cos\theta \, \mathrm{d}\theta \, \mathrm{d}A = \pi I_0 \ . \qquad \text{(SA.15)}$$

For a cone with half-aperture angle ϕ, this becomes

$$\begin{aligned} &P = \pi I_0 \sin^2\phi \\ \Rightarrow &P \propto I_0 \sin^2\phi \ , \end{aligned} \qquad \text{(SA.16)}$$

where $\sin\phi$ is equal to the numerical aperture. Therefore, if the numerical aperture of the point source is known, the fraction of light that can be coupled into the fiber can be calculated.

Assuming that maximum transfer of energy is required, adapting the numerical aperture of the Lambertian source to the fiber core gives an angle of $\phi = 14°$.

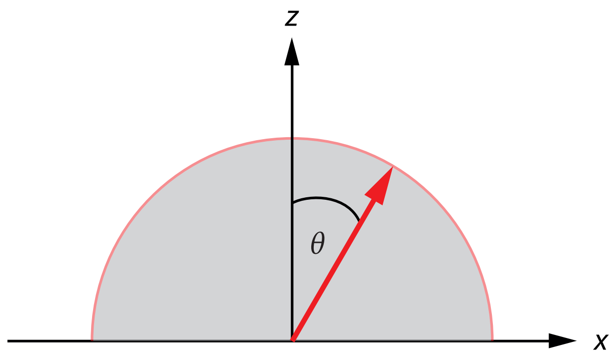


Figure SA.5 Emission characteristics of a Lambertian radiator. The emitted power (represented by the height of the red arrow) to any direction determined by the angle θ.

With Eq. (SA.16) this leads to a total radiant power for the numerical aperture of

$$P \propto I_0 \sin^2 14^\circ$$
$$\Rightarrow P \propto 0.058\, I_0 \ .$$

Hence, 6% of the light emitted by the active region of the LED can be coupled into the fiber. If the volume between fiber and GaAs is filled with water, the effective numerical aperture increases due to the higher refractive index of water compared to air. This, in turn, leads to a higher percentage of light coupling.

2. The maximum acceptance angle of a fiber in air is given by (Figure SA.6)

$$a_{\max} = \arcsin \sqrt{n_c^2 - n_{cl}^2} \ . \qquad \text{(SA.17)}$$

With $n_c = n_{\text{silica}} = 1.46$ and $a_{\max} = 14^\circ$, we obtain

$$n_{cl} = \sqrt{n_c^2 - \sin^2 14^\circ} \approx 1.44 \ .$$

3. The etendue or throughput characterizes the amount of light which passes through the system. It is determined by the area of the pupil times the solid angle subtended by the light source as seen from the pupil. In the above scenario, the pupil was considered to be the core of the fiber with an area of π and the solid angle being the square of the numerical aperture. The etendue of the Lambertian source was thus proportional to the square of the numerical aperture. For any arbitrary system, the etendue is proportional to the square of the Helmholtz-Lagrange invariant as given in Eq. (A.31). This is directly related to the Helmholtz-Lagrange invariant, which is

$$H = y \cdot \text{NA} = y' \cdot \text{NA}'$$

for a paraxial system. The low fraction of light coupled into the fiber is caused by the etendue of the fiber being not taken into consideration. When two optical

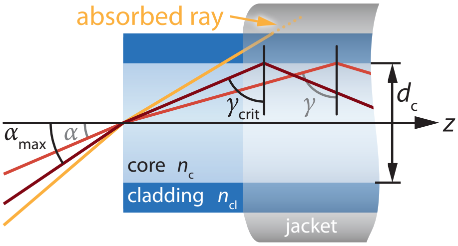


Figure SA.6 Geometry of GaAs LED-fiber coupling. In our case, we have $\alpha_{\max} = 14°$.

components are combined, their etendues must be equal to maximize the amount of transferred light. This can be facilitated by using a common pupil-like lens to match the etendue of the source and the fiber. This is mostly implemented in practice to achieve a higher coupling efficiency.

PA.6
Point-spread function and Strehl ratio

In practice, the Strehl ratio (or definition brightness) is defined as the normalized ratio of the point-spread intensity on the axis for the real system to the ideal system, that is,

$$\mathcal{S} = \frac{I_{\mathrm{PSF}}^{\mathrm{real}}(0,0)}{I_{\mathrm{PSF}}^{\mathrm{ideal}}(0,0)} \quad .$$

This relation is used to characterize diffraction-limited optical systems. Write the Strehl ratio for (Fraunhofer) far field conditions by assuming a circular, evenly illuminated pupil as a function of the wave aberration $\mathcal{W}(x_{\mathrm{P}}, y_{\mathrm{P}})$.

Solution:

The point-spread function (PSF) expressed in terms of the Fraunhofer diffraction integral (see [14]) is given by

$$\mathrm{PSF}(x', y') = \frac{i \exp(-ikz)}{\lambda\,|z|} \cdot \exp\left[\frac{ik}{2z} \cdot (x^2 + y^2)\right] \cdot \iint U(x_{\mathrm{P}}, y_{\mathrm{P}}) \cdot \exp\left[-\frac{ik}{z} \cdot (x^2 + y^2)\right] \mathrm{d}x_{\mathrm{P}} \mathrm{d}y_{\mathrm{P}} \qquad \text{(SA.18)}$$

with the complex pupil function $U(x_\mathrm{p}, y_\mathrm{p})$. For an evenly illuminated pupil on the optical axis, we have

$$\mathrm{PSF}(0,0) = \frac{i\exp(-ikz)}{\lambda\,|z|}\,U_0\cdot\iint \exp(ik\cdot\mathcal{W}(x_\mathrm{p},y_\mathrm{p}))\mathrm{d}x_\mathrm{p}\mathrm{d}y_\mathrm{p}\ , \qquad \text{(SA.19)}$$

where $\mathcal{W}(x_\mathrm{p}, y_\mathrm{p})$ is the wave abberation. With this result, the Strehl ratio is given by

$$\begin{aligned}\mathcal{S} =& \frac{I_\mathrm{PSF}^\mathrm{real}(0,0)}{I_\mathrm{PSF}^\mathrm{ideal}(0,0)} = \frac{\left|\iint \exp(ik\cdot\mathcal{W}(x_\mathrm{p},y_\mathrm{p}))\mathrm{d}x_\mathrm{p}\mathrm{d}y_\mathrm{p}\right|^2}{\left|\iint \mathrm{d}x_\mathrm{p}\mathrm{d}y_\mathrm{p}\right|^2}\\ =& \frac{1}{A_\mathrm{pupil}^2}\left|\iint \exp(ik\cdot\mathcal{W}(x_\mathrm{p},y_\mathrm{p}))\mathrm{d}x_\mathrm{p}\mathrm{d}y_\mathrm{p}\right|^2\ . \qquad \text{(SA.20)}\end{aligned}$$

One can easily see the relationship between Strehl ratio and the wavefront aberration.

PA.7
Taylor expansion of wavefront aberrations

Show that only terms with the products h^2, r^2, and $hr\cos\alpha$ are physically relevant for the Taylor expansion of the wavefront aberration function $\mathcal{W}(h, r, \alpha)$.

Solution:

Considering Eq. (A39) in the book, which expands the wave aberration function in a power series in polar coordinates, it can be seen that $\mathcal{W}$ depends on h, r, α. Assuming the system to be rotationally symmetric, some parameters remain unchanged upon rotation (rotational invariance). The squared parameter of the object height h^2, the pupil height r^2, and the scalar product $\mathbf{h}\cdot\mathbf{r} = hr\cos\theta$ remain unchanged. Hence, all terms of the power series expansion must be integral powers of these invariants for rotational symmetry to be satisfied. Only these invariant terms are thus required for the Taylor expansion of wave aberration function.

A more thorough analysis starts from Eq. (A39) and its full expansion

$$\mathcal{W}(h,r,\alpha) = \sum_{l,n,m} \mathcal{W}_\mathrm{lnm} h^\mathrm{l} r^\mathrm{n} \cos^\mathrm{m}\alpha$$

and takes rational symmetries into account, e.g.,

$$\mathcal{W}(h,r,\alpha) = \mathcal{W}(-h,r,\alpha)\ .$$

PA.8
Zernike expansion of wavefront aberrations

According to Zernike, the wave aberrations for defocus and spherical aberrations are defined in normalized coordinates as

$$\mathcal{W}_{\text{def}}(r) = c_2^0\,(2r^2 - 1)\ ,$$
$$\mathcal{W}_{\text{sph}}(r) = c_4^0\,(6r^4 - 6r^2 + 1)\ .$$

1. For the spherical aberration across a circular pupil, calculate the peak-valley value, the mean value, and the RMS_{wfe} value for $c_4^0 = 1$.
2. When determining the Zernike coefficient, one always assumes a normalized pupil radius. This reference plays a key role in the result. In reality, the size of the pupil often cannot be determined very accurately. How does the determination of a spherical Zernike coefficient change if the assumed pupil size deviates by 5%? What is the associated error in the specification of the defocus?
3. Factorize an aspherical cylinder surface into Zernike polynomials using the equation $F(x,y) = y^4$.

Solution:

1. The wave abberation is given by

$$\mathcal{W}_{\text{sph}}(r) = 6r^4 - 6r^2 + 1 \qquad \text{(SA.21)}$$

   a) **Peak-valley value**

$$\text{Derivative}: \ \frac{\mathrm{d}\mathcal{W}_{\text{sph}}}{\mathrm{d}r} = 24r^3 - 12r$$
$$\text{Extrema}: \ \frac{\mathrm{d}\mathcal{W}_{\text{sph}}}{\mathrm{d}r} = 0 : r_1 = 0;\ \mathcal{W}(r_1) = 1;\ r_2 = \frac{1}{\sqrt{2}};\ \mathcal{W}(r_2) = -\frac{1}{2}$$
$$\text{Boundary value}: \ r_3 = 1;\ \ \mathcal{W}(r_3) = 1$$

   As the extrema are -0.5 and $+1$, the peak-valley value is $\mathcal{W}_{p-v} = 1.5$.

b) **Mean value**

$$\overline{\mathcal{W}} = \frac{\int_0^1 \mathcal{W}_{\text{sph}}(r) 2\pi r \, \mathrm{d}r}{\int_0^1 2\pi r \, \mathrm{d}r}$$

$$= \frac{1}{\pi} \cdot \int_0^1 \left(6r^4 - 6r^2 + 1\right) 2\pi r \, \mathrm{d}r = 2 \int_0^1 (6r^5 - 6r^3 + r) \, \mathrm{d}r$$

$$= 2 \cdot \left(r^6 - \frac{3}{2}r^4 + \frac{1}{2}r^2\right)_0^1 = 2 \cdot \left(1 - \frac{3}{2} + \frac{1}{2}\right)$$

$$= 0$$

The mean value of the aberration is thus zero.

c) **RMS value**

$$\mathcal{W}_{\text{RMS}}^2 = \frac{\int_0^1 \left(\mathcal{W}_{\text{sph}}(r) - \overline{\mathcal{W}}\right)^2 2\pi r \, \mathrm{d}r}{\int_0^1 2\pi r \, dr}$$

$$= \frac{1}{\pi} \cdot \int_0^1 \left(6r^4 - 6r^2 - 1\right)^2 2\pi r \, \mathrm{d}r$$

$$= 2 \cdot \left[\frac{36}{10}r^{10} - 9r^8 + 8r^6 - \frac{3}{2}r^4 - 3r^4 + \frac{1}{2}r^2\right]_0^1 = \frac{1}{5}$$

$$\Rightarrow \mathcal{W}_{\text{RMS}} = \frac{1}{\sqrt{5}} = 0.447$$

2. We take the expression for the aperture error

$$\mathcal{W}_{\text{sph}}(\overline{r}) = \overline{c}_4^0 \cdot (6\overline{r}^4 - 6\overline{r}^2 + 1) \quad ,$$

and insert a normalized radius $\overline{r} = \varepsilon \cdot r$ with $\varepsilon = 0.95$, which expresses the uncertainty in the radius. With this approach, it follows that

$$\begin{aligned} \mathcal{W}_{\text{sph}}(\overline{r}) &= \overline{c}_4^0 \cdot (6\overline{r}^4 - 6\overline{r}^2 + 1) \\ &= \overline{c}_4^0 \cdot (6\varepsilon^4 r^4 - 6\varepsilon^2 r^2 + 1) \quad . \end{aligned} \tag{SA.22}$$

Comparing Eq. (SA.22) to the regular expression

$$\mathcal{W}(r) = c_4^0 \cdot (6r^4 - 6r^2 + 1) + c_2^0 \cdot (2r^2 - 1)$$

for exponents of r^4 and r^2 leads to

$$r^4 : \overline{c}_4^0 = \frac{c_4^0}{\varepsilon^4} \;, \quad \frac{\Delta c_4^0}{c_4^0} = \frac{c_4^0 - \overline{c}_4^0}{c_4^0} = 1 - \frac{1}{\varepsilon^4} = -0.288$$

$$r^2 : \frac{c_2^0}{c_4^0} = 3 \cdot \left(\frac{\varepsilon^2 - 1}{\varepsilon^2}\right) = -0.324 \quad .$$

The error in the Zernike coefficient of spherical aberration is thus approximately 20. This demonstrates the sensitive impact of the correct pupil radius in an impressive manner.

3. Here, we do not use the representation of the Zernike polynomials (as given in Table A.3), but their equivalent representation in x-y-coordinates as given in [2], Table 11-1. The relevant polynomials up to y^4 as the largest exponent are

$\mathcal{Z}_4^{-4}(x,y)$	$= y^4 + x^4 - 6x^2y^2$	quatrefoil
$\mathcal{Z}_4^{-2}(x,y)$	$= 4y^4 - 4x^4 - 4x^2y^2 + 3x^2 - 3y^2$	secondary astigmatism
$\mathcal{Z}_4^{0}(x,y)$	$= 6y^4 + 6x^4 + 12x^2y^2 - 6x^2 - 6y^2 + 1$	spherical aberration

We now need to make the terms that include x^4 and x^2y^2 disappear simultaneously, which leads to a condition for these three polynomials.
In the process, terms with exponents of lower-order like x^2, y^2, and absolute terms are generated which, in turn, need to be compensated by lower-order polynomials.

In accordance, the following Zernike polynomials are also required to generate the desired surface shape:

$\mathcal{Z}_2^{-2}(x,y)$	$= y^2 - x^2$	astigmatism
$\mathcal{Z}_2^{0}(x,y)$	$= 2x^2 + 2y^2 - 1$	defocus
$\mathcal{Z}_0^{0}(x,y)$	$= 1$	piston, constant offset

To generate the surface, we use the following composition:

$$F(x,y) = y^4$$
$$= c_4^{-4}\mathcal{Z}_4^{-4} + c_4^0\mathcal{Z}_4^0 + c_4^{-2}\mathcal{Z}_4^{-2} + c_2^{-2}\mathcal{Z}_2^{-2} + c_2^0\mathcal{Z}_2^0 + c_0^0\mathcal{Z}_0^0 \ .$$

Inserting the polynomials and comparing the coefficients of the same order yields a system of linear equations given by

$$y^4 : c_4^{-4} + 6c_4^0 + 4c_4^{-2} = 1 \qquad \text{(SA.23)}$$
$$x^4 : c_4^{-4} + 6c_4^0 + 4c_4^{-2} = 0 \qquad \text{(SA.24)}$$
$$x^2y^2 : -6c_4^{-4} + 12c_4^0 - 4c_4^{-2} = 0 \qquad \text{(SA.25)}$$
$$x^2 : -6c_4^0 + 3c_4^{-2} - c_2^{-2} + 2c_2^0 = 0 \qquad \text{(SA.26)}$$
$$y^2 : -6c_4^0 - 3c_4^{-2} + c_2^{-2} + 2c_2^0 = 0 \qquad \text{(SA.27)}$$
$$1 : c_4^0 - c_2^0 + c_0^0 = 0 \ . \qquad \text{(SA.28)}$$

The system of equations can be solved based on the linear dependence and compensation effect of the higher-order exponents. Equations (SA.23) – (SA.25) can be solved separately. The results obtained supply the necessary coefficients to solve Eqs. (SA.26) – (SA.28). We finally obtain

c_4^{-4}	$= -\frac{1}{2}$	quatrefoil
c_4^{-2}	$= \frac{1}{2}$	secondary astigmatism
c_4^{0}	$= \frac{5}{12}$	spherical aberration
c_2^{-2}	$= \frac{3}{2}$	astigmatism
c_2^{0}	$= \frac{5}{4}$	defocus
c_0^{0}	$= \frac{5}{6}$	offset

Thus, a cylindrical aspheric surface determined by $F(x, y) = y^4$ generates astigmatism and spherical aberration.

PA.9
Brewster angle

> Derive the equation for the Brewster angle γ_B from Eqs. (A63) – (A66).

Solution:

According to Figure (A.30), we only have an *s*-component in the reflected light. Consequently, Eq. (A65) becomes zero, which is equivalent to

$$\tan(\gamma + \gamma') \to \infty \quad .$$

This is, in turn, equivalent to

$$\begin{aligned} \gamma + \gamma' &= 90^\circ \quad \text{or} \\ \gamma' &= 90^\circ - \gamma \quad . \end{aligned}$$

With Snell's law given in Eq. (A2), we can write

$$\begin{aligned} n \sin\gamma &= n' \sin\gamma' \\ \Rightarrow \frac{\sin\gamma}{\sin(90^\circ - \gamma)} &= \frac{\sin\gamma}{\cos\gamma} = \tan\gamma = \frac{n'}{n} \quad . \end{aligned} \tag{SA.29}$$

From Eq. (SA.29), we finally obtain

$$\gamma_\mathrm{B} = \arctan\left(\frac{n'}{n}\right) \quad .$$

PA.10
Gaussian Beams

ASTRA is a television satellite which travels in a geostationary orbit (distance to sea level $h = 35.8\,\text{km}$). The signal is transmitted via radiation with a wavelength of $\lambda = 2.7\,\text{cm}$ and a transmission power of $P = 100\,\text{W}$. The radiation can be considered as a Gaussian beam with the spatial envelope function (A80).

1. Show that the emitted beam fulfills the paraxial approximation.
2. Calculate power and intensity received by a parabolic antenna (diameter $d = 1\,\text{m}$) located at sea level. *Hint:* Use the approximation $1 - \mathrm{e}^{-x} \approx x$.
3. What would happen if the satellite could transmit the digital television signal with visible light (e.g., by using an argon ion laser)?

Solution:

1. The geostationary orbit is at a distance of $h = 35.8\,\text{km}$ from sea level ($h = 0$). The antenna has a diameter of

$$D_\mathrm{S} = 2w_0 = 2\ \text{m} \ .$$

The Rayleigh length is obtained by using Eq. (A83), $\nu = 11\,\text{GHz}$, and $\lambda = 2.7\,\text{cm}$. Hence, we have

$$z_\mathrm{R} = \frac{\pi w_0^2}{\lambda} = \frac{\pi D_\mathrm{S}^2}{4\lambda} = 116.4\ \text{m} \ .$$

Since the antenna diameter is much bigger than the wavelength, and as the Rayleigh length is much bigger than the antenna diameter, it is allowed to use the paraxial approximation.

2. At a receiver on Earth, the beam diameter for $h \gg z_\mathrm{R}$ is given by (see also Eq. (A86))

$$\begin{aligned} 2w_\mathrm{E} &= 2w_0 \cdot \sqrt{1 + \left(\frac{h}{z_\mathrm{R}}\right)^2} \\ &\approx 2w_0 \cdot \frac{h}{z_\mathrm{R}} = 615\ \text{km} \ . \end{aligned}$$

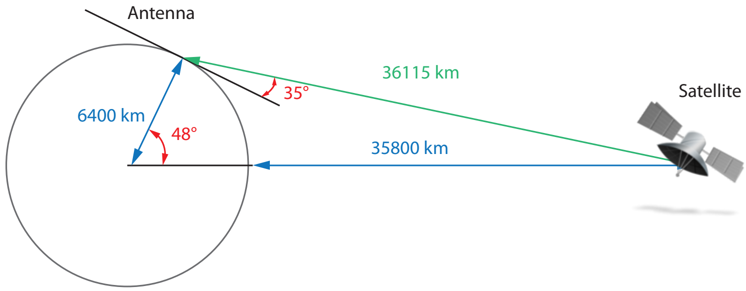


Figure SA.7 Geometry of the ASTRA satellite travelling is a geostationary orbit.

The total received power is obtained by integration over the entire Gaussian profile of the receiver antenna area. It is thus given by

$$\begin{aligned} P_\mathrm{E} &= \int_0^{D_\mathrm{E}/2} P_\mathrm{S} \cdot \exp\left[-2\left(\frac{r}{w_\mathrm{E}}\right)^2\right] 2\pi r \,\mathrm{d}r \\ &= P_\mathrm{S} \cdot \left[1 - \exp\left(-2\left(\frac{D_\mathrm{E}}{2w_\mathrm{E}}\right)^2\right)\right] \\ &\approx 2P_\mathrm{S} \cdot \left(\frac{D_\mathrm{E}}{2w_\mathrm{E}}\right)^2 = 100\ \mathrm{W} \cdot 2 \cdot 2.28 \times 10^{-12} \\ &= 0.53\ \mathrm{nW} \quad . \end{aligned}$$

Here, we used the approximation $1 - \mathrm{e}^{-x} \approx x$, as $D_\mathrm{E} \ll w_\mathrm{E}$. The emitted intensity is related to the power via

$$\begin{aligned} P_\mathrm{S} &= I_0 \frac{\pi w_\mathrm{S}^2}{2} \quad , \\ I_0 &= \frac{2P_\mathrm{S}}{\pi w_\mathrm{S}^2} = \frac{2P_\mathrm{S}}{\pi (D_\mathrm{S}/2)^2} = \frac{8P_\mathrm{S}}{\pi D_\mathrm{S}^2} \\ \Rightarrow I &= \frac{I_0}{1 + (h/z_0)^2} = \frac{8P_\mathrm{S}}{\pi D_\mathrm{S}^2} \cdot \left(\frac{1}{1 + (h/z_0)^2}\right) = 6.7 \times 10^{-10}\ \mathrm{Wm}^{-2} \quad . \end{aligned} \tag{SA.30}$$

Note that the above calculation is only valid for a receiver centered on the equator. In Karlsruhe, in southern Germany (latitude of 48°N), the receiving antenna must be mounted at an angle of about 35° versus the vertical. Due to the oblique projection onto the surface of the Earth, a much bigger area is covered by the antenna than the calculated 615 km, but of course also at lower powers.

3. For the case of a green laser ($\lambda = 514$ µm), we can also use Eq. (SA.30) and obtain

$$z_\mathrm{R} = \frac{\pi w_0^2}{\lambda} = \frac{\pi D_\mathrm{S}^2}{4\lambda} = 6280 \text{ km} \;,$$

$$2w_\mathrm{E} = 2w_0 \cdot \sqrt{1 + \left(\frac{h}{z_\mathrm{R}}\right)^2} = 11.6 \text{ m} \;,$$

$$P_\mathrm{E} = P_\mathrm{S} \left[1 - \exp\left(-2\left(\frac{D_\mathrm{E}}{2w_\mathrm{E}}\right)^2\right)\right] = 1.49 \text{ W} \;,$$

$$I = \frac{8P_\mathrm{S}}{\pi D_\mathrm{S}^2} \cdot \frac{1}{1 + (h/z_\mathrm{R})^2} \approx \frac{8P_\mathrm{S}}{\pi D_\mathrm{S}^2} \cdot \left(\frac{z_\mathrm{R}}{h}\right)^2 = 1.9 \text{ Wm}^{-2} \;.$$

Thus, the received power is much higher. However, we only reach one household in Karlsruhe.
In reality, one would use much smaller antennas for the communication of satellites with laser light. Due to $z_\mathrm{R} \propto D_\mathrm{S}^2$, this would lead to shorter Rayleigh lengths and with the square broader spot diameters on Earth. A laser source on the satellite with a spot diameter of 2 cm would thus have to a spot diameter of about 120 km on Earth.

PA.11
Autocorrelation function, spectral density, and coherence length

Calculate the autocorrelation function $\mathcal{G}$, the spectral density $\sigma(\omega)$, and coherence length L_c for various pulse forms and spectral distributions:
1. Gaussian pulse
2. Rectangle pulse
3. Lorentz spectrum
4. sech^2 pulse

Use the definitions for $\mathcal{G}(\Delta t)$ and $\sigma(\omega)$ as given in Eqs. (A111) and (A109), respectively. For the coherence length as well as the spectral density use appropriate definitions, such as the full width at half maximum (FWHM), second momentum of a normalized function, or second momentum of a squared normalized function.

Solution:

Please refer to Problem P7.2.