Chapters 1-3

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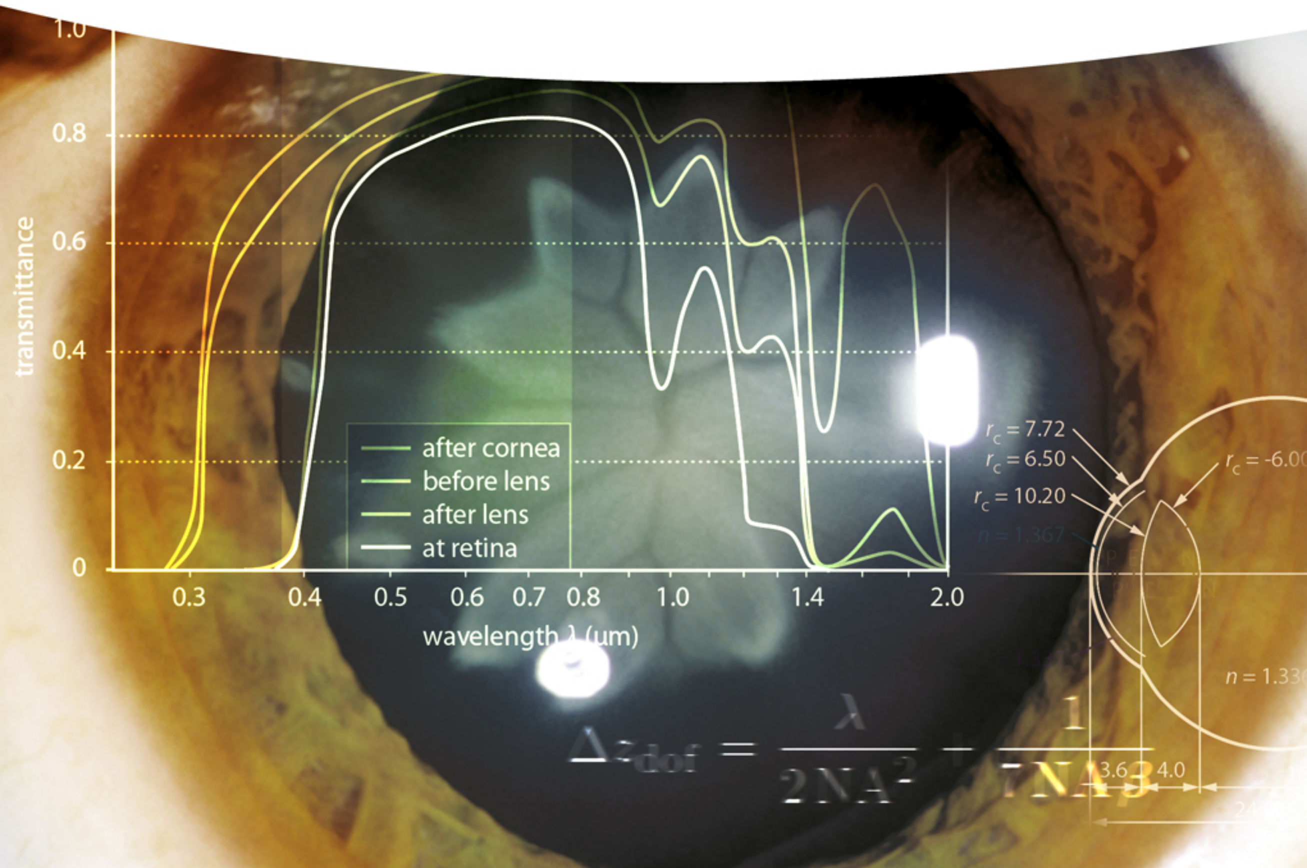
Structure and Function / Optics of the Human Eye / Visual Disorders and Major Eye Diseases



Solutions to Problems in „Optical Devices in Ophthalmology and Optometry"

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PI.1
Size of the retinal image

The retinal image size can be calculated via

$$|h_{\mathrm{I}}| = \kappa \cdot \overline{\mathrm{N'I'_0}} \tag{3.7}$$

$$|h_{\mathrm{I}}| = |h_0| \cdot \frac{\overline{\mathrm{N'I'_0}}}{\overline{\mathrm{O_0N}}} \ , \tag{3.8}$$

where $\kappa = -h_{\mathrm{O}}/\overline{\mathrm{O_0N}}$ (Section 2.1.3). For an object which is located at infinity (relaxed eye) and with a mean refractive power of the eye of 60 D, we may derive the relation

$$|h_{\mathrm{I}}| \approx 16.667\,\mathrm{mm} \cdot \kappa \ . \tag{3.9}$$

If κ is given in degrees, the relation can also be expressed by:

$$|h_{\mathrm{I}}| \approx 0.291\,\mathrm{mm} \cdot \kappa^{\circ} \tag{3.10}$$

with $\kappa = -h_{\mathrm{O}}/\overline{\mathrm{O_0V}}$ (Figure 2.3).

1. Please derive Eq. (3.9) or (3.10).
2. Calculate the size of the retinal image for a tower (50 m high at a distance of 1 km), a person (1.8 m high at a distance of 10 m), a thumbnail (diameter of 2 cm at a distance of 60 cm), and for the full moon ($\kappa = 0.5°$).

Solution:

1. From Eq. (3.7), $|h_{\mathrm{I}}| = \kappa \cdot \overline{\mathrm{N'I'_0}}$ is given. When the object is far away (relaxed eye/far vision), according to Figure 2.3 we can approximate

$$\overline{\mathrm{N'I'_0}} \approx \overline{\mathrm{N'F'}} \ . \tag{SI.1}$$

Then, from Eqs. (2.6) and (2.3), we conclude that

$$\overline{\mathrm{N'F'}} = \overline{\mathrm{FP}} = \frac{1}{\mathcal{D}^2_{\mathrm{eye}}} \ . \tag{SI.2}$$

The mean refractive power of a human eye is 60 D, so $\mathcal{D}_{\mathrm{eye}} = 1/f' = 60$ D, which means that

$$f' = \frac{1}{60\ \mathrm{m}^{-1}} \approx 16.667\ \mathrm{mm}.$$

Substituting Eqs. (SI.2) into (SI.1), we find

$$\overline{\mathrm{N'I'_0}} \approx f' \ . \tag{S1.3}$$

Accordingly, we have for small angles κ :

$$|h_{\mathrm{I}}| = 16.667\ (\mathrm{mm}) \cdot \kappa\ (\mathrm{rad})\ \ . \tag{3.9}$$

Converting radians into degrees leads to

$$1^\circ = \frac{\pi}{180}\ \mathrm{rad}, \ \frac{1^\circ}{1\ \mathrm{rad}} = \frac{\pi}{180} \approx 0.01745\ \ .$$

Equation (SI.1) can thus also be written as

$$|h_{\mathrm{I}}| = 16.667\ (\mathrm{mm}) \cdot (\kappa^0 \cdot 0.01745) \approx 0.291\ \mathrm{mm} \cdot \kappa^\circ. \tag{3.10}$$

2. Tower: height of $50\ \mathrm{m}$, distance of $1000\ \mathrm{m}$. Hence, we obtain

$$\arctan\left(\frac{\text{height}}{\text{distance}}\right) = \arctan\frac{50}{1000} \approx 0.05\ \mathrm{rad}$$

and thus $|h_{\mathrm{I}}| = 16.67\mathrm{mm} \cdot 0.05\ \mathrm{rad} \approx 0.83\mathrm{mm}$.

Person: height of $1.8\ \mathrm{m}$, distance of $10\ \mathrm{m}$. Hence, we obtain

$$\arctan\left(\frac{\text{height}}{\text{distance}}\right) = \arctan\frac{1.8}{10} \approx 0.178\ \mathrm{rad}$$

and thus $|h_{\mathrm{I}}| = 16.67\ \mathrm{mm} \cdot 0.1781\ \mathrm{rad} \approx 2.7\ \mathrm{mm}$.

Thumb: diameter of $2\mathrm{cm}$, distance $60\ \mathrm{cm}$. Hence, we obtain

$$\arctan\left(\frac{\text{diameter}}{\text{distance}}\right) = \arctan\frac{0.02}{0.6} \approx 0.033\ \mathrm{rad}$$

and thus $|h_{\mathrm{I}}| = 16.67\ \mathrm{mm} \cdot 0.033\ \mathrm{rad} \approx 0.56\ \mathrm{mm}$.

Moon: $\kappa = 0.5^\circ$, $|h_{\mathrm{I}}| = 0.291 \cdot \kappa^0 \approx 0.145\ \mathrm{mm}$. The moon thus appears roughly as a third in size as compared to the thumb.

PI.2
Gullstrand Eye model

Calculate the position of the eye's entrance pupil relative to the corneal vertex. Please do also calculate the diameter of the entrance pupil relative to the iris aperture by using the Gullstrand Eye model #1.

Solution:

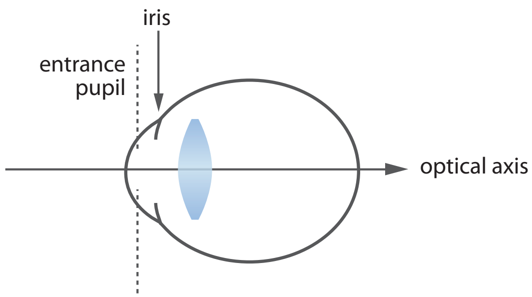

Figure SI.1 Position of iris and entrance pupil relative to the lens

The entrance pupil is the image of the iris by the cornea. From Figure 2.13, the following parameters are given:

$$n' = 1, \ s = -3.6 \text{ mm}, \ n = 1.336 \ .$$

We know from Eq. (2.25) that the total refractive power of the cornea amounts to $\mathcal{D}'_c$ = 43.06 D. With the modified imaging equation (A14), we can write

$$\frac{n'}{s'} - \frac{n}{s} = \frac{n'}{f'} = \mathcal{D}'_c \ ,$$
$$\frac{1}{s'} - \frac{1.336}{-3.6 \text{ mm}} = 43.06 \text{ D},$$

which leads to $s' = -3.049$ mm. Because of the negative sign, both the object and the image are "behind" the cornea. The entrance pupil is then located at 3.05 mm behind the corneal vertex.
The angular magnification (A15) is given by

$$\beta = \frac{s'n}{sn'} = \frac{-3.049 \cdot 1.336}{-3.6} \approx 1.13 \ .$$

Assuming that d' is the diameter of the entrance pupil and d is the diameter of the iris, it follows that

$$\beta = \frac{d'}{d} \rightarrow d' \Rightarrow \beta \cdot d = 1.13 \ d \ .$$

Thus, the diameter of entrance pupil is about 13 % larger than the iris diameter.

PI.3
Reflectance of the cornea

Calculate the reflectance of the cornea at the vertex with the Fresnel equation (A4).

Solution:

The Fresnel equation (A4) is derived from the more general Fresnel equation (A65) for the special case of a normal incident ray ($\gamma = 0°$) and neglected absorption by

$$R = \left(\frac{n - n'}{n + n'}\right)^2 .$$

For the cornea, we have $n = 1$ and $n' = 1.376$ according to the Exact Gullstrand Eye # 1 model shown in Figure 2.13. Thus, we have

$$R = \left(\frac{n - n'}{n + n'}\right)^2 = \left(\frac{1 - 1.376}{1 + 1.376}\right)^2 = 0.025.$$

The reflectance of the cornea at the vertex for a normal incident ray is therefore 2.5%.

PI.4
Radius of curvature

Variations of the corneal radius of curvature Δr_C mean a change of the corneal refractive power $\Delta \mathcal{D}$ which can be calculated via

$$\Delta \mathcal{D} \approx -\frac{\Delta r_\mathrm{C} \cdot \mathcal{D}_\mathrm{c}'^2}{n_\mathrm{c} - 1} \tag{3.11}$$

where $\mathcal{D}_\mathrm{c}'$ is the refractive power of the corneal front surface and n_c the refractive index of the cornea.

a) Please derive Eq. (3.11).
b) Verify the following statement for an emmetropic Gullstand Eye: A variation of the corneal radius of curvature by ± 0.1 mm changes the eye's refractive power by approximately ± 0.6 D.

Solution:

1. In the paraxial approximation according to Eq. (2.22), the refractive power of a spherical surface is

$$\mathcal{D}' = \frac{n' - n}{r_\mathrm{C}} \,, \tag{SI.4}$$

where n and n' are the refractive indices of the media on the incident and refracted side, respectively. Taking the derivative of Eq. (SI.4) with respect to r_C, we obtain

$$\frac{\mathrm{d}\mathcal{D}'}{\mathrm{d}r_\mathrm{C}} = -\frac{n' - n}{r_\mathrm{C}^2}, \text{ or in approximation } \Delta\mathcal{D}' = -\frac{\Delta r_\mathrm{C} \cdot (n' - n)}{r_\mathrm{C}^2} \quad . \tag{SI.5}$$

Re-arranging Eq. (SI.4), yields

$$\frac{1}{r_\mathrm{C}} = \frac{\mathcal{D}'}{n' - n} \quad . \tag{SI.6}$$

By substituting Eq. (SI.5) into Eq. (SI.6), we obtain

$$\Delta\mathcal{D}' = -\frac{\Delta r_\mathrm{C} \cdot \mathcal{D}'^2}{n' - n} \quad . \tag{SI.7}$$

In the case of the cornea, where $\mathcal{D}' = \mathcal{D}'_\mathrm{c}$ (refractive power of the corneal front surface), $n' = n_\mathrm{c}$ (refractive index of cornea), $r = r_\mathrm{C}$ (curvature of cornea), and n = 1 (refractive index of air), we finally get

$$\Delta\mathcal{D}' \approx -\frac{\Delta r_\mathrm{C} \cdot \mathcal{D}'^2_\mathrm{c}}{n_\mathrm{c} - 1} \quad . \tag{3.11}$$

2. We can calculate the refractive power of the corneal front surface by using Eq. (2.23) which leads to $\Delta\mathcal{D}'_\mathrm{c} = 48.33$ D.
Using the given data $\Delta r_\mathrm{C} = \pm 0.1$ mm and n_c = 1.376, it follows that

$$\Delta\mathcal{D}' \approx -\frac{\Delta r_\mathrm{C} \cdot D'^2_\mathrm{c}}{n_\mathrm{c} - 1} = -\frac{(\pm 10^{-4}\mathrm{m}) \cdot (48.83\mathrm{m}^{-1})^2}{1.376 - 1} = \mp 0.63 \text{ D} \quad .$$

Thus, a variation of the corneal radius of curvature by $\pm$ 0.1 mm changes the eye's refraction (refractive power) by approximately $\mp$ 0.6 D. An increase of the radius of curvature by 0.1 mm thus reduces the refractive power by more than a half a diopter, that is, an emmetropic eye becomes hyperopic.

PI.5
Eye length

A small variation on an emmetropic eye's axial length ΔL_{eye} with a refractive power of $\mathcal{D}'_{\text{eye}}$ means a change of refraction by $\Delta\mathcal{D}$. We may approximate this change with

$$\Delta A_{\text{far}} = -\Delta\mathcal{D}' \approx -\frac{\Delta L_{\text{eye}} \cdot \mathcal{D}'_{\text{eye}}{}^2}{n}. \tag{3.12}$$

1. Please derive Eq. (3.12).
2. Verify the following statement for an emmetropic Gullstrand Eye with $\mathcal{D}'_{\text{eye}}$ = 60 D and n = 1.336: The variation of the eye length by ± 0.37 mm changes the eye's refraction by approximately ∓ 1 D.

Solution:

1. According to the lens maker's equation (A14), we have

$$\frac{n'}{s'} - \frac{n}{s} = \frac{n'}{f'} \ . \tag{SI.8}$$

Applying this formula to the eye's optical system, we get

$$\frac{n'}{L_{\text{eye}}} - A_{\text{far}} = \mathcal{D}'_{\text{eye}} \tag{SI.9}$$

in which $\mathcal{D}'_{\text{eye}}$ is the total refractive power of the eye and $A_{\text{far}} = n/s = 1/s$ the far point refraction. L_{eye} denotes the eye length for the emmetropic eye. A change in L_{eye} by a small amount ΔL_{eye} results in a change of the far point refraction (i.e., the refractive status of the eye) by ΔA_{far}. Therefore, the eye's refractive power is given by

$$\frac{n'}{L_{\text{eye}} + \Delta L_{\text{eye}}} - (A_{\text{far}} + \Delta A_{\text{far}}) = \mathcal{D}'_{\text{eye}} \ ,$$

from which we can easily obtain

$$\begin{aligned}\Delta A_{\text{far}} &= \frac{n'}{L_{\text{eye}} + \Delta L_{\text{eye}}} - (A_{\text{far}} - \mathcal{D}'_{\text{eye}}) \\ &= \frac{n'}{L_{\text{eye}} + \Delta L_{\text{eye}}} - \frac{n'}{L_{\text{eye}}} = -\frac{n' \cdot \Delta L_{\text{eye}}}{(L_{\text{eye}} + \Delta L_{\text{eye}}) \ \cdot L_{\text{eye}}} \ ,\end{aligned}$$

where we used Eq. (SI.9).
As we assume small changes of the eye length ($\Delta L_{\text{eye}} \ll L_{\text{eye}}$), we may use the following approximation:

$$\Delta A_{\text{far}} \approx -\frac{n' \cdot \Delta L_{\text{eye}}}{L_{\text{eye}}^2} \ . \tag{SI.10}$$

In the case of an emmetropic eye (A_{far} = 0), we obtain from Eq. (SI.9)

$$L_{\text{eye}} = \frac{n'}{D'_{\text{eye}}} \tag{SI.11}$$

Inserting Eq. (SI.11) into Eq. (SI.10), we finally obtain

$$\Delta A_{\text{far}} \approx -\Delta L_{\text{eye}} \frac{\mathcal{D}'^{\,2}_{\text{eye}}}{n'} \tag{SI.12}$$

An increase of the eye length of an emmetropic eye leads to a negative refractive change, that is, the eye becomes myopic. A decrease of the eye length of an emmetropic eye leads to a positive refractive change, that is, that the eye becomes hyperopic.

2. Substituting given data $\mathcal{D}'_{\text{eye}}$ = 60 D, n = 1.336 and ΔL_{eye} = ± 0.37 mm, we find

$$\Delta A_{\text{far}} \approx -\frac{\mathcal{D}'^{\,2}_{\text{eye}} \Delta L_{\text{eye}}}{n'} = -\frac{(60\text{ D})^2 \cdot (\pm 0.37\text{ mm})}{1.336} \approx \mp 0.997 \approx \mp 1$$

Thus, a variation of the eye length by ± 0.37 mm changes the eye's refraction by approximately ∓ 1 D.

PI.6
Stereoscopic vision

In order to check the stereoscopic vision of an eye, real and virtual test objects are used.

1. A real test object shall be used up to a stereo angle of $\varepsilon = 5''$. What minimum stereoscopic depth perception ΔL_{min} must this object have if it is viewed from a distance of 5 m and the interpupillary distance is PD = 62 mm.
2. A virtual stereoscopic test object consists of two identical test objects T_l and T_r (e.g., stripes or triangles). These objects are horizontally arranged (distance Δy = 20 mm) above or below a central focus object F (e.g., circle). The three objects T_l, T_r and F lie all in one test plane which is perpendicular to the viewing direction and located at a distance of 5 m. An optical system ensures that each eye of the patient can only see one test object. In this regard, we can distinguish between the following cases:
   - symmetric allocation, that is, T_l (T_r) is seen by the left (right) eye, and
   - asymmetric allocation, that is, T_l (T_r) is seen by the right (left) eye.

   Due to the small relative shift of both identical images on the retina, the patient perceives a virtual object T which seems to float behind and in front of the test plane at a distance ΔL (virtual stereoscopic effect).

   a) Is the normal stereoscopic resolution sufficient to have a three-dimensional impression?
   b) At what distance behind and in front of the test plane does a patient (with normal stereoscopic vision) see the test objects (interpupillary distance PD = 65 mm)?
   c) Which allocation do we have to choose, if the patient shall perceive a floating test object located in front of the test plane?
   d) Calculate the relative local shift Δs of the retinal images for an eye with a refractive power of 60 D.

Solution:

1. We use the geometry of Figure 2.11 with a stereo angle of $\varepsilon = 5'' = 2.424 \times 10^{-5}$rad, the viewing distance $L = 5\,\text{m}$ and the interpupillary distance PD $=$ 62 mm. According to Eq. (2.18), the minimum stereoscopic depth perception with a small stereo angle is

$$\Delta L_{\text{min}} = \frac{\varepsilon L^2}{\text{PD}} \quad .$$

Inserting the given data into Eq. (2.18) leads to

$$\Delta L_{\text{min}} = \frac{\varepsilon L^2}{\text{PD}} = \frac{5'' \cdot (5\text{ m})^2}{62\text{ mm}} = \frac{2.424 \times 10^{-5}\text{ rad} \cdot (5\text{ m})^2}{62\text{ mm}} = 9.77\text{ mm} \quad .$$

The test object thus has to provide a depth structure of about 1 cm to become resolved by the naked eye from a 5 m distance.

2. a) Using again Eq. (2.11) we set $\Delta y = s_\mathrm{p} = 20$ mm and $L = 5$ m. Inserting the given data into Eq. (2.15), we get a stereo angle of

$$\varepsilon = 2\ \mathrm{arctan}\left(\frac{s_\mathrm{p}}{2L}\right) \approx 0.004\ \mathrm{rad} \approx 0.23^\circ = 13.8' \ .$$

This minimum stereo angle determines the smallest angle that can be resolved by the eye and still allows stereoscopic perception. Under appropriate conditions, the human eye has a minimum stereo angle of $\varepsilon_\mathrm{min} = 10''$. In our case, the calculated minimum stereo angle of $13.8'$ is much greater than $10''$. Therefore, it is possible to have a 3D impression.

b) The test person can see the test objects at ΔL_b behind the test plane and at ΔL_f in front of the test plane. ΔL_b and ΔL_f are given by Eqs. (2.16) and (2.17) as

$$\Delta L_\mathrm{f} = \frac{L \cdot s_\mathrm{p}}{\mathrm{PD} + s_\mathrm{p}} \ ,$$
$$\Delta L_\mathrm{b} = \frac{L \cdot s_\mathrm{p}}{\mathrm{PD} - s_\mathrm{p}} \ .$$

With $s_\mathrm{p} = 20$ nm, $L = 5$ m and PD $= 65$ mm, we have

$$\Delta L_\mathrm{f} = \frac{L \cdot s_\mathrm{p}}{\mathrm{PD} + s_\mathrm{p}} = \frac{5\ \mathrm{m} \cdot 20\ \mathrm{mm}}{65\ \mathrm{mm} + 20\ \mathrm{mm}} \approx 1.18\ \mathrm{m} \ ,$$
$$\Delta L_\mathrm{b} = \frac{L \cdot s_\mathrm{p}}{\mathrm{PD} + s_\mathrm{p}} = \frac{5\ \mathrm{m} \cdot 20\ \mathrm{mm}}{65\ \mathrm{mm} - 20\ \mathrm{mm}} \approx 2.22\ \mathrm{m} \ .$$

Hence, in this case, the test person (with normal stereoscopic vision) sees the test objects 1.18 m in front of the test plane or 2.22 m behind it.

c) Considering Figure 2.11, the patient shall perceive a floating test object located in front of the test plane with asymmetric allocation.

d) From exercise I.1, we can use $|h_\mathrm{I}|$ = 16.667 (mm) with $\kappa = \varepsilon$. It follows that

$$\varepsilon = 2\ \mathrm{arctan}\left(\frac{s_\mathrm{p}}{2L}\right) \approx 0.004\ \mathrm{rad}$$
$$\Rightarrow |h_\mathrm{I}| = 16.667\ \mathrm{mm} \cdot 0.004\ \mathrm{rad} \approx 67\ \mu\mathrm{m} \ .$$

Therefore, the relative lateral shift $\Delta s = |h_\mathrm{I}|$ of the retinal images is about 67 µm for an eye with a refractive power of 60 D.

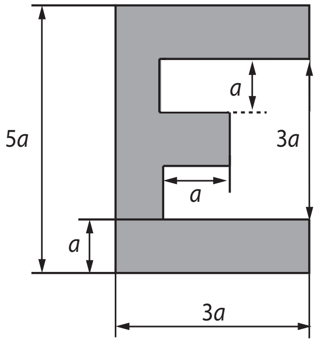

Figure SI.2 Standard letter E used to determine the refraction of the eye.

PI.7
Resolving power of the eye

1. In order to determine the refraction of an eye, the standard letter "E" is placed at a distance of 2 ft (= 6096 mm) from the eye. Calculate the size of the letter for a visual acuity of $V = 1$.
2. The retinal image resembles the image shown in the Figure 3.19. Calculate the image size on the retina and compare the result to the distance of the retinal cones (in the fovea).
3. Sometime in the future, we will be visited by aliens from the planet XIR2050 whose star emits light only in the red and infrared spectral range and whose atmosphere allows only near-infrared light (wavelengths between 1 and 1.5 μm) to pass through. The eyes of the aliens are adapted to these conditions and the aliens' visual acuity is similar to that of our eyes. How will they fare on Earth? If you were to be selected to travel to XIR2050, how would you prepare for your visit? What should you expect to be faced with on this planet?

Solution:

1. The standard letter E has a height of $\gamma = 5a$ whereby a is the bar width that characterizes the resolution limit. Using the distance $s = 6096$ mm and the critical angle of resolution of $\alpha = 1'$, one obtains Using the distance $s = 6096$ mm and the critical angle of resolution of $\alpha = 1'$, one obtains

$$a = \alpha \cdot s$$

$$y = 5a = 5s\alpha = 5 \cdot 6096 \text{ mm} \cdot 1' = 5 \cdot 6096 \text{ mm} \cdot \frac{\pi}{180 \cdot 60} = 8.86 \text{ mm} \ .$$

2. Applying the data from Table 2.1 and using the effective focal length of the eye

$$f_{\text{eff}} = \overline{\text{N}'\text{F}'} = 24.385 - 7.331 \approx 17.1 \text{ mm} ,$$

we obtain the size of the retinal image of the letter E

$$\gamma' = 5 f_{\text{eff}} \alpha = 25 \text{ µm} .$$

Accordingly, the scale is $a = \gamma'/5 = 5$ µm. This is equal to the Airy diameter in Eq. (A77) given by

$$d_{\text{Airy}} = 1.22 \cdot \frac{\lambda f_{\text{eff}}}{d_{\text{iris}}} = 5.6 \text{ µm}$$

for a pupil diameter of approximately $d_{\text{iris}} = 4$ mm and a focal length of the eye of $f_{\text{eff}} = 17$ mm. This corresponds to Rayleigh's criterion of resolution.
In the fovea, the density of cones is maximum with 140,000 mm^{-2}. This means that each cone (assuming for simplicity a squared cross section) takes up an area of

$$A = \frac{1}{n} = 7.1 \times 10^{-6} \text{ mm}^2 = 7.1 \text{ µm}^2 .$$

This corresponds to the length of an edge or a distance between cones of $\Delta x = 2.7$ µm. Accordingly, about 2 cones are present per length a at the resolution limit, which is quite consistent with the sampling theorem.

3. Assuming the resolution limit to be an angle of $1'$ or the Airy diameter (A77), as was done above, it follows that

$$a = f\alpha = d_{\text{Airy}} = 2.44 \cdot \frac{\lambda f}{d_{\text{iris}}}$$
$$\Rightarrow \alpha = 2.44 \cdot \frac{\lambda}{d_{\text{iris}}} = \text{const.}$$

In order for the critical angle to remain constant while the mean wavelength is approximately $1.3 \text{ µm}/0.55 \text{ µm} = 2.4\times$ larger, the eye pupil of the alien must be larger by the same factor. Accordingly, the adaptation of the aliens to the conditions on their planet leads to a pupil of approx. 9.6 mm for normal vision.
Because of the spectral adaptation of their eyes to the atmospheric transmittance on their planet, the aliens only see in the near-infrared (NIR) spectral range on Earth. Since the atmosphere of the Earth has a fairly good transmittance in this range, objects on Earth are illuminated well between 1 µm and 1.5 µm by sunlight or daylight and the aliens can see well.
In turn, on the planet XIR2050, the atmosphere allows only the light of their star above 1 µm to pass through. However, the rods and cones of human eyes are not sensitive in this spectral range. This means that we would actually see nothing. Travelling to the planet would be useless unless one would use a strong flashlamp or a NIR-sensitive camera and a head-mounted imaging display for transformation of the NIR radiation on XIR2050 to the visible spectrum. This is equivalent to using NIR imaging devices of the type known from nightvision goggles.

PI.8
Refracting errors

1. In the case of cataracts, it used to be common in the past to simply perforate the turbid lens and sort of remove it in a surgical process. The eye was rendered aphakic (i.e., left without eye lens). Where is the image of a far point in an aphakic eye? Would it have been possible to help this person with spectacles? How strong would these spectacles have to have been?
2. Is it true that myopic people see small things better? How much of a difference is there as compared to a person with normal vision?
3. What is your comment regarding the assertion that myopic people can see sharp images under water; meaning that they do not need a pair of goggles?
4. In diving schools, it is taught that you see objects under water 33% bigger and 25% closer. Prove, if this is really true.

Solution:

1. The refractive power of the aphakic eye is exclusively based on the effect of the cornea, which is given by $\mathcal{D}'_\mathrm{c}$ = 43.06 D (Table 2.1). This refractive power acts right on the vertex of the cornea located at a distance d = 24.385 mm from the retina in the normal eye (Figure 2.13)[1].

   Using $n' = 1.336$ (Table 2.1), that is, the refractive index of the aqueous humor (vitreous), the image of an object placed at infinity is located at a distance

$$s' = \frac{n'}{\mathcal{D}'_\mathrm{c}} = 31.0 \text{ mm} \tag{SI.13}$$

   from the corneal vertex. As the eye length is 24.4 mm (Table 2.1), the image plane is located about 6.6 mm behind the retina, which means that an aphakic eye is extremely hyperopic.
   The refractive power in the vertex plane of the cornea to correct for the extracted lens would have to be

$$\mathcal{D}'_\mathrm{corr} = \frac{n'}{d} = \frac{1.336}{24.385} \text{ mm} = 54.78 \text{ D}.$$

   As the corneal power given by $\mathcal{D}'_\mathrm{c}$ = 43.06 D, one would have to add 11.7 D, e.g., by increasing the corneal curvature, which is beyond practical limits (see Problem

1) Here we assumed that the cornea can be considered as a thin lens with the principal planes at the front surface. In reality, the meniscus-shaped corneal structure with the distribution of positive and negative refractive power at the front and the rear side, respectively, has a principal plane which is moved -0.45 mm left to the rear side. This nearly corresponds to the position of the front vertex. Therefore, the assumption above is quite accurate.

P10.13).
Placing spectacles with a refractive power of $\mathcal{D}'_s$ at a distance $L_c = 16$ mm in front of the eye would result in a refractive power of

$$\mathcal{D}'_{corr} = \mathcal{D}'_c + \mathcal{D}'_s - L_c \cdot \mathcal{D}'_c \mathcal{D}'_s \ , \tag{SI.14}$$

that is, the sum of the refractive powers of two thin lenses at a distance L_c. Equation (SI.14) is often called Gullstrand's Equation. The spectacle lens then must have a refractive power of

$$\mathcal{D}'_s = \frac{\mathcal{D}'_{corr} - \mathcal{D}'_c}{1 - L_c \cdot \mathcal{D}'_c} = \frac{54.78 \text{ D} - 43.06 \text{ D}}{1 - 0.016 \text{ m} \cdot 43.06 \text{ D}} \approx 37.7 \text{ D} \ . \tag{SI.15}$$

This means that strongly positive spectacles would have to be used. However, the refractive power can be reduced somewhat by placing the spectacles closer to the eye (see table below). However, there are obvious limits for such thick lenses. The patients would only be able to view objects within a very small angle ("tunnel vision").

α (mm)	$\mathcal{D}'_s$ (D)
12	24.3
10	20.6

2. In the case of myopia, an excess of refractive power (by action of cornea and lens) exists compared to an emmetropic eye with the same axial eye length (so-called *refractive myopia*). In another situation, we may compare two eyes, one emmetropic eye with a length of L_{emm} and one with identical refractive (corneal and lens) power but with a length of $L_{myo} > L_{emm}$ (so-called *axial myopia*).
   a) First, we compare the image magnification for an uncorrected myopic person to an emmetropic person (in both myopic situations).
   From the lens equations (A14) and (A15), we easily obtain for the image magnification of the myopic eye

$$\beta_{myo} = \frac{s'_{myo}}{s}$$

   and for the emmetropic eye

$$\beta_{emm} = \frac{s'_{emm}}{s} \ .$$

   We assume for both cases the observed object to be located in the far point s of the myopic eye. Then, by definition we have

$$s = \frac{1}{A_{far,myo}}$$

with the far point refraction on the myopic eye $A_{\text{far,myo}}$. The magnification then can be written as

$$|\beta_{\text{myo}}| = \frac{s'_{\text{myo}}}{s} = A_{\text{far,myo}} \cdot s'_{\text{myo}}$$

and

$$|\beta_{\text{emm}}| = \frac{s'_{\text{emm}}}{s} = A_{\text{far,myo}} \cdot s'_{\text{emm}} \quad .$$

To obtain sharp images on the retina of an object in the far point of the myopic eye, the following conditions must be fulfilled

$$s'_{\text{myo}} = L_{\text{myo}}$$

and

$$s'_{\text{emm}} = L_{\text{emm}} \quad .$$

For the *myopic eye*, this condition is fulfilled per definition via

$$s'_{\text{myo}} = \frac{n'}{\mathcal{D}_{\text{myo}}} = L_{\text{myo}}$$

with $\mathcal{D}_{\text{myo}}$ being the total refractive power of the myopic eye. In contrast, the *emmetropic eye* has to accommodate in order to perceive a sharp image of an object located at $s = 1/A_{\text{far,myo}}$. This increases the refractive power of the emmetropic eye by the factor $\Delta\mathcal{D}_{\text{acc}}$. After adequate accommodation, we thus get

$$s'_{\text{emm}} = \frac{n'}{\mathcal{D}_{\text{myo}} + \Delta\mathcal{D}_{\text{acc}}} = L_{\text{emm}}$$

and finally for the magnifications

$$|\beta_{\text{myo}}| = A_{\text{far,myo}} \cdot L_{\text{myo}} \quad ,$$
$$|\beta_{\text{emm}}| = A_{\text{far,myo}} \cdot L_{\text{emm}} \quad .$$

In the case of so-called refractive myopia, we have $L_{\text{myo}} = L_{\text{emm}}$ and consequently identical magnifications.
In the case of so-called axial myopia, we have $L_{\text{myo}} > L_{\text{emm}}$. Consequently, the myopic eye will see an object located in the far point of the myopic eye which appears slightly larger than an object located at the same position seen by an adequately accommodated emmetropic eye.

b) Before we can compare the image magnifications, we calculate how to correct the myopic eye. To this end, we use the optical diagram in Figure S.3 for the corrected eye (and, for simplicity, assume the lenses to be thin).
The far point Q_{far} of the myopic eye is located at a finite distance $s_{\text{far}} = 1/A_{\text{far}}$ in front of the eye. In order to correct a myopic eye, the spectacles must

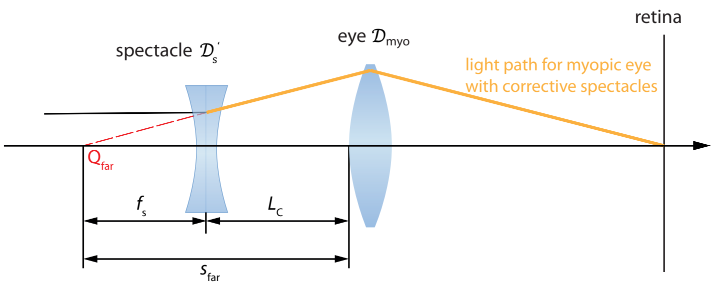

Figure SI.3 Refractive error correction with spectacles

have a negative refractive power. The resulting optical system comprising the spectacles and the myopic eye (reduced in its effect to a thin lens in air having a refractive power of D_{myo}) can be described as follows:
L_{c} is the always positive distance of the corrective lens (here assumed to be extremely thin) to the corneal vertex (see also Figure 5.30) and s_{far} the far point distance of the eye. According to Figure SI.3 (note that s_{far} is negative in the case of myopia!), we can write

$$f_s = s_{\text{far}} + L_{\text{c}} \quad .$$

With the far point refraction $A_{\text{far}} = 1/s_{\text{far}}$ we derive for the refractive power of the spectacles

$$\mathcal{D}_{\text{s}}'(L_{\text{c}}) = \frac{1}{f_s} = \frac{A_{\text{far}}}{1 + L_{\text{c}} \cdot A_{\text{far}}} \quad . \tag{SI.16}$$

For a myopic eye, A_{far} can also be written as

$$A_{\text{far}} = \mathcal{D}_{\text{ideal}} - \mathcal{D}_{\text{myo}} \ , \tag{SI.17}$$

with $\mathcal{D}_{\text{ideal}}$ being the refractive power of the ideal eye (emmetropic eye with the same axial eye length) and $\mathcal{D}_{\text{myo}}$ being the refractive power of the myopic eye. Note again that for a myopic eye, A_{far} is negative, since the far point is situated in front of the eye ($s_{\text{far}} < 0$) and consequently $\mathcal{D}_{\text{myo}} > \mathcal{D}_{\text{ideal}}$.
The resulting refractive power of the corrected myopic eye obtained by the addition of two refractive elements (spectacle lens, myopic eye) being separated by a distance L_{c} (see for derivation Section A.1.3) is given by

$$\mathcal{D}_{\text{tot}} = \mathcal{D}_{\text{myo}} + \mathcal{D}_{\text{s}}'(L_{\text{c}}) - L_{\text{c}} \cdot \mathcal{D}_{\text{myo}} \cdot \mathcal{D}_{\text{s}}'(L_{\text{c}}) \quad . \tag{SI.18}$$

Substituting Eq. (SI.16) into Eq. (SI.18) yields

$$\mathcal{D}_{\text{tot}} = \mathcal{D}_{\text{myo}} + \frac{A_{\text{far}}}{1 + L_{\text{c}} \cdot A_{\text{far}}} - \frac{L_{\text{c}} \cdot \mathcal{D}_{\text{myo}} \cdot A_{\text{far}}}{1 + L_{\text{c}} \cdot A_{\text{far}}} \quad .$$

Using Eq. (SI.17) to substitute $\mathcal{D}_{\rm myo}$, we find

$$\mathcal{D}_{\rm tot} = \frac{\mathcal{D}_{\rm ideal}}{1 + L_{\rm c} \cdot A_{\rm far}} \quad . \tag{SI.19}$$

For example, if the far point is at a distance of 0.2 m in front of the myopic eye and the spectacle lens is placed at 16 mm in front of the corneal vertex, we find

$$\mathcal{D}'_{\rm s}(L_{\rm c}) = \frac{A_{\rm far}}{1 + L_{\rm c} \cdot A_{\rm far}} \approx -5.4 \text{ D}$$

and

$$\mathcal{D}_{\rm tot} = \frac{\mathcal{D}_{\rm ideal}}{1 + L_c \cdot A_{\rm far}} = \frac{59\text{D}}{1 - 0.016 \text{ m} \cdot 1 \text{ D}} \approx 63.7 \text{ D} \quad . \tag{SI.20}$$

c) Next, we want to compare the image magnification for a corrected myopic eye in comparison to an emmetropic eye. We define the spectacle magnification as

$$m_{\rm s} = \frac{\text{retinal image size of myopic eye with correction}}{\text{retinal image size of an emmetropic eye}} \quad . \tag{SI.21}$$

Essentially, we compare two sharp retinal images. For a corrected myopic eye, the retinal image size of a distant object with an angular size κ is given by

$$h'_{\rm myo,corr} = \frac{\kappa}{\mathcal{D}_{\rm tot}} \quad .$$

The corresponding retinal image size of an emmetropic eye is given by

$$h'_{\rm emm} = \frac{\kappa}{\mathcal{D}_{\rm ideal}} \quad .$$

Thus, Eq. (SI.21) becomes

$$\begin{aligned} m_{\rm s} &= \frac{\text{retinal image size of myopic eye with correction}}{\text{retinal image size of an emmetropic eye}} \\ &= \frac{h'_{\rm myo,corr}}{h'_{\rm emm}} = \frac{\mathcal{D}_{\rm ideal}}{\mathcal{D}_{\rm tot}} \quad . \end{aligned}$$

With Eqs. (SI.17) and (SI.19), we finally obtain

$$m_{\rm s} = \frac{1}{1 + L_{\rm c} \cdot A_{\rm far}} \quad . \tag{SI.22}$$

For a spectacle position of $L_{\rm c}$ = 16 mm in front of the cornea, we calculate the following values for various myopic refractions:

A_{far}	$\mathcal{D}'_{\text{s}}$	m_{s}
-1	-1.0	0.98
-2	-2.1	0.96
-3	-3.2	0.95
-5	-5.4	0.92
-10	-11.9	0.84

Evidently, a corrected myopic person indeed sees objects smaller when viewing through the correcting spectacles in comparison to an emmetropic person.

d) Finally, we want to compare the image magnification for a corrected myopic eye with an uncorrected myopic eye. For this purpose, we define the spectacle magnification using the symbols in Figure SI.4 as

$$m'_{\text{s}} = \frac{\text{retinal image size of myopic eye with correction}}{\text{retinal image size of a myopic eye without correction}} = \frac{\text{angular size with correction}}{\text{angular site without correction}} = \frac{\alpha_{\text{corr}}}{\alpha} \; . \tag{SI.23}$$

The uncorrected myopic eye views an object with height y at a distance x from the entrance pupil. The object appears under an angle $\alpha = y/x$.

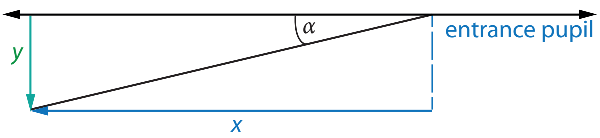

Figure SI.4a Image formation for an uncorrected eye

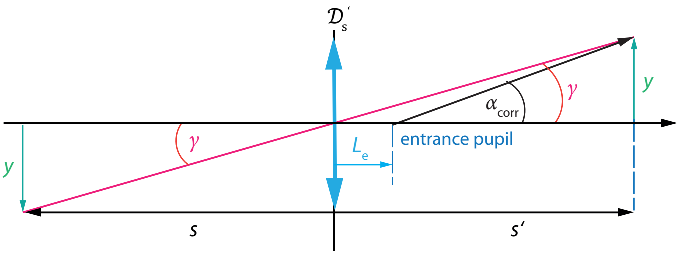

Figure SI.4b Image formation by a correcting lens

Now, we put a lens with a refractive power of $\mathcal{D}'_{\text{s}}$ in front of the eye. The object should be at a distance s from the lens. The object now subtends an angle γ.

From Figure SI.4b, we can deduce

$$y' = y \cdot \frac{s'}{s}$$

and

$$\alpha_{\text{corr}} = \frac{y'}{s' - L_{\text{e}}}$$

in which we can substitute y' and where L_{e} is the distance of the lens from the entrance pupil plane of the myopic eye. With Eq. (SI.23), we find

$$m'_{\text{s}} = \frac{\alpha_{\text{corr}}}{\alpha} = \frac{s'x}{s(s' - L_{\text{e}})} = \frac{sx}{1 - L_{\text{e}}S'} \quad ,$$

in which we used $S = s^{-1}$ and $S' = s'^{-1}$. Further, we can write the image equation (A14) for the situation of Figure SI.4 as

$$S' = S + \mathcal{D}_{\text{s}} \quad .$$

If we now assume a distant object, for which $x \to \infty,\ S \to 0,\ x \cdot S \to 1,\ S' \to \mathcal{D}'_s$, we finally obtain

$$m'_{\text{s}} = \frac{1}{1 - L_{\text{e}} \cdot \mathcal{D}_{\text{s}}} \quad .$$

Using $\mathcal{D}_{\text{ideal}} = 59\,\text{D}$ and $L_{\text{e}} = L_{\text{c}} + 3\text{mm} = 19\,\text{mm}$, we find the values shown in the following Table for the spectacle magnification as a function of the degree of short-sightedness A_{far}:

A_{far}	$\mathcal{D}'_{\text{s}}$	m'_{s}
-1	-1.0	0.98
-2	-2.1	0.96
-3	-3.2	0.94
-5	-5.4	0.91
-10	-11.9	0.82

Evidently, a myopic eye indeed sees objects correspondingly smaller when viewing through the correcting spectacles (but at least sharply). Moreover, we recognize that only in the case of a mild refraction error, the power of the spectacles is approximately equal to the refractive error.

3. Assuming, (for approximation purposes,) the refractive index of water to be $n = 1.334$, the refractive effect of the front surface of the cornea ($\mathcal{D}'_{\text{a}}$= 48.8 D; see Eq. (2.23)) is almost not effective under water, since there is only a small difference in refractive index at the front of the eye.

Using Eq. (2.23) and data from the Exact Gullstrand Eye # 1 (Table 2.1), we get for the refractive power of the corneal front surface in water

$$\mathcal{D}'_{\mathrm{a}} = \frac{1.375 - 1.334}{0.0077} = 5.45\ \mathrm{D}\ .$$

According to Eq. (2.24), we get for the refractive power of the corneal back surface

$$\mathcal{D}'_{\mathrm{p}} = \frac{1.336 - 1.376}{0.0068} = -5.88\ \mathrm{D}\ .$$

With Eq. (2.20), the total refractive power of the cornea in water follows as

$$\begin{aligned} D'_{\mathrm{c,in\ water}} &= 5.45\ \mathrm{D} + (-5.88)\ \mathrm{D} - \frac{0.0005}{1.336}(5.45 \cdot (-5.88)\ \mathrm{D} \\ &= -0.42\ \mathrm{D} \approx\ 0\ \mathrm{D}\ . \end{aligned}$$

Thus, only the refractive power $\mathcal{D}_{\mathrm{l}} = 19.1\,\mathrm{D}$ of the lens (relaxed) or $\mathcal{D}_{\mathrm{l}} = 33.1\,\mathrm{D}$ (accommodated) remains in this case.
According to Table 2.1 (Exact Gullstrand Eye #1), an emmetropic eye has a total refractive power of about 59 D in air (relaxed vision). In water, the power is reduced by about $40\,\mathrm{D}\ \approx\ 59\,\mathrm{D} - 19\,\mathrm{D}$ and if compensated by maximum accommodation still by about $26\,\mathrm{D}\ \approx\ 59\,\mathrm{D} - 33\,\mathrm{D}$. As the eye length is constant, this power reduction leads to significant (under-water) hyperopia. The same is true for a myopic eye. However, the (under-water) hyperopia is reduced by the refraction (refractive error) in air, if the refractive error originates from axial myopia (i.e., a longer eye length). In addition, in order to see a noticeable (under-water) hyperopia reduction effect, the (in-air) myopia must be extremely high (above $-20\,\mathrm{D}$). There is no difference at all between a myopic and an emmetropic person's vision under water if the myopia in air is caused by a smaller radius of curvature of the cornea.

4. As we know from Eq. (A15), the magnification is given by

$$\beta = \frac{ns'}{n's}\ .$$

In this case, s', n' and s are constant. The magnification in air can thus be written as

$$\beta_{\mathrm{air}} = \frac{s'}{n's}$$

and under water as

$$\beta_{\mathrm{w}} = \frac{1.33s'}{n's}\ .$$

Thus, when s (the distance between the object and the people's eyes) is constant, people see objects under water in comparison to air as

$$\frac{\beta_{\mathrm{w}}}{\beta_{\mathrm{air}}} = 1.33\ ,$$

that is 33% bigger.

When people see an object showing the same apparent size under water as in air, the magnification is

$$\beta_{\text{air}} = \beta_{\text{w}} \quad .$$

Accordingly,

$$\frac{n_{\text{air}} s'}{n' s_{\text{air}}} = \frac{n_{\text{w}} s'}{n' s_{\text{w}}} \quad .$$

Due to the structure of eyes, s' and n' are always constant. Consequently, $n_{\text{air}}/s_{\text{air}}$ should be equal to $n_{\text{w}}/s_{\text{w}}$.

Substituting n_{air} = 1 and n_{w} = 1.33, we obtain

$$s_{\text{w}} = 1.33 \cdot s_{\text{air}} \;\text{ or }\; s_{\text{air}} = 0.75 \cdot s_{\text{w}} \quad .$$

Thus, an object appearing at the same size under water and in air, seems to be 25% closer in water than in air.

PI.9
Refractive errors

1. Use the Gullstrand Eye model to calculate the power of spectacles needed to correct an eye if the far point is located 45 cm in front of the eye. Because of the frame of spectacles, the glasses are placed at a distance of 15 mm in front of the corneal vertex.
2. The same spectacle glasses as in 1) have inadvertently been mounted in a frame so that we now have a distance from the vertex of only 10 mm. Does this improve or worsen the correction of the refractive error?
3. Draw a conclusion from 2) regarding how a contact lens would have to be designed.

Solution:

1. The far point s_{far} is located 45 cm in front of the eye. This corresponds to a far point refraction (refractive error) of

$$A_{\text{far}} = \frac{1}{s_{\text{far}}} = \frac{1}{-45\text{cm}} \approx -2.22\text{ D} \quad .$$

According to Problem PI.8, the required back vertex power of the corrective spectacle glass is given by

$$\mathcal{D}'_{\text{s}}(L_{\text{c}}) = \frac{A_{\text{far}}}{1 + L_{\text{c}} \cdot A_{\text{far}}} \quad .$$

As the spectacles are placed at a vertex distance of L_c = 15 mm from the cornea, we have

$$\mathcal{D}'_s(L_c) = \frac{-2.22}{1 + 0.015(-2.22)} = -2.3 \text{ D} \ .$$

2. If the distance of the spectacles from the corneal vertex is only $L_c = 10$ mm, the correction results as

$$\mathcal{D}'_s(L_c) = \frac{-2.22}{1 + 0.01(-2.22)} = -2.27 \text{ D} \ .$$

Hence, the correction is slightly poorer. However, this will hardly be noticeable. The sign is disadvantageous, though, since the additional error cannot be corrected by means of accommodation.

3. For a contact lens of negligible thickness, we have $L_c \approx 0$. Thus, the required back vertex power of the contact lens is almost equal to the refractive error (≈ -2.22 D) of the eye.

PI.10
Chromatic aberrations

> The eye shows some notable chromatic aberration of almost 2 D in the visible spectral range. Why do we generally not notice this, whereas an optical instrument (e.g. photo camera) with similar chromatic aberration would be unusable? Could the different width of the blue-white-red stripes of the French flag (Tricolore) have anything to do with this?

Solution:

An axial chromatic error of 2 D between red and blue is clearly above the bothersome limit of a refraction error (the limit is at about 0.25 D). However, the corresponding color receptors (*L*, *S*, *M* cones) are rather elongated in shape and somewhat staggered in their depth. Moreover, these types of blur are usually corrected by the brain based on experience. Therefore, the effect of longitudinal chromatic aberration in the eye is less bothersome than in technical systems for imaging.

The difference in axial depth of the colored images of 2 D causes a difference in focal intercept (Figure 6.11) of

$$\Delta s = s_{\text{red}} - s_{\text{blue}} = \frac{1}{59.5} \text{ m} - \frac{1}{61.5} \text{ m} = 0.55 \text{ mm} \ .$$

Theoretically, this defocussing causes a chromatic magnification difference (lateral chromatic aberration, see Section A.1.9) of

$$\frac{\Delta y'}{y'} = \frac{\Delta s}{f} = 60.5 \text{ mm}^{-1} \cdot 0.55 \text{ mm} = 3.3\% \ .$$

This magnitude is already noticeable by eye. However, in reality, the circumstances are far more complicated:

1. The principal nodal points vary by color. The estimation presented above is thus simplified.
2. The retina is a curved "receiver" which has an impact on the geometric conditions.
3. Usually, extensive image fields are viewed with a rotating eye in order to optimally utilize the focusing range of the fovea. Accordingly, the field angle produced in the eye is not very large, and the difference in chromatic magnification is relatively small.
4. For finite field angles, we have rod receptors in the peripheral range. These are not color-sensitive.

Accordingly, the apparent difference in the width of the stripes of the Tricolore is most likely not only a consequence of the lateral chromatic aberration of the eye.

PI.11
Stereo-camera system

Let us consider the special case of a stereo-camera system. Both cameras have equal parameters (in particular equal focal lengths f), the image planes of the two cameras are co-planar, and the x axes of the image planes are parallel to the baseline (Figure 3.20). The distance between the two cameras is $b = 200\,\mathrm{cm}$. The cameras use $1/2''$ CCD chips with 1024 horizontal pixels. A camera system system of this type shall be used to track objects (e.g. surgical instruments) in a volume (a^3 = 50 x 50 x 50 cm^3) around a patient's head from a distance $z \approx 3\,\mathrm{m}$.

1. Calculate the stereo-disparity (difference between x_L and x_R) for various objective focal lengths f.
2. Which objective focal length would you recommend to achieve a tracking volume of maximum size or maximum point accuracy?
3. Calculate the maximum attainable (lateral) resolution of the system at the optimal objective focal length. What causes the resolution to be less in reality? How can the resolution be increased?
4. Compare this resolution to that of an acoustical tracking system with a frequency of 50 kHz. Which phase measuring accuracy must be at least attained with an ultrasound system?

Solution:

In Figure 3.20, f is the focal length, x the lateral object coordinate. In this case, we have the middle of the camera axes as the origin.

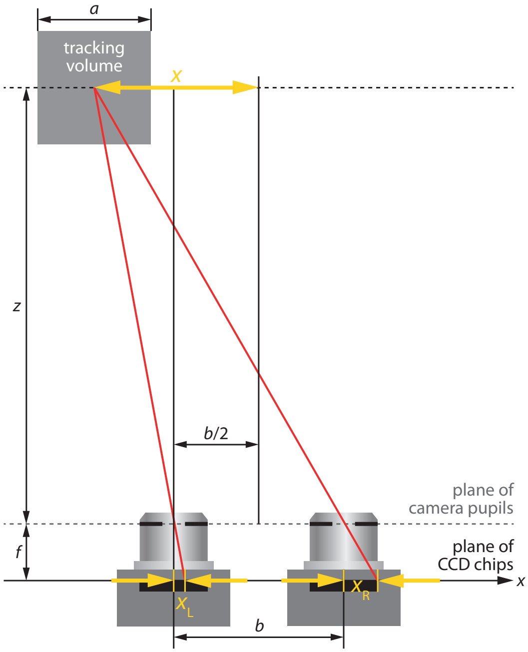

Figure 3.20 Geometry of a stereo-camera system discussed in Problem I.??.

1. According to the theorem of Thales, the following equations apply to similar triangles:

$$\frac{x_\mathrm{L}}{f} = \frac{x - b/2}{z} \quad ,$$

$$\frac{x_\mathrm{R}}{f} = \frac{x + b/2}{z} \quad .$$

Accordingly, the mean of the stereodisparity of the center of the volume is

$$\Delta x = x_\mathrm{R} - x_\mathrm{L} = \frac{f \cdot b}{z} \quad . \tag{SI.25}$$

Using $b = 200$ cm and $z = 300$ cm, the following examples can be calculated:

1)	$f = 10$ mm $= 1$ cm	$\Rightarrow \Delta x = 0.67$ cm
2)	$f = 20$ mm $= 2$ cm	$\Rightarrow \Delta x = 1.33$ cm
3)	$f = 30$ mm $= 3$ cm	$\Rightarrow \Delta x = 3.3$ cm

2. Considering just one camera (i.e., the camera on the right), we obtain – similar to the preceding exercise – for the outer point of the measuring volume at a distance $x = a/2$:

$$\frac{x_\mathrm{R,max}}{f} = \frac{b/2 + a/2}{z}$$

from which follows that

$$a = \frac{2x_{\mathrm{R,max}} \cdot z}{f} - b \ . \tag{SI.26}$$

The maximum permissible camera coordinate $x_{\mathrm{R,max}}$ is determined by the size of the sensor.
According to Eq. (SI.26), a increases with decreasing f. As a consequence, a large transverse measuring volume can be obtained with small focal lengths. Conversely, according to Eq. (SI.25), a large focal length results in the transverse disparity being large, which results in high accuracy.

3. The optimal focal length is the largest focal length that still permits the measuring range of the parameter a. This mean that, for $x = a/2$, the transverse disparity is equal to half the chip width $x_{\mathrm{R,max}} = D$, that is, $x_{\mathrm{R,max}} = D/2$. Accordingly, the optical focal length is given by Eq. (SI.26) as

$$f = \frac{2x_{\mathrm{R,max}} \cdot z}{a + b} = \frac{z \cdot D}{a + b} = 15.2 \text{ mm} \ , \tag{SI.27}$$

with $z = 300\,\mathrm{cm}$, $b = 200\,\mathrm{cm}$, $a = 50\,\mathrm{cm}$ and $D = \frac{1}{2}'' = 1.27\,\mathrm{cm}$.
From

$$\frac{x_{\mathrm{R}}}{f} = \frac{x + \frac{b}{2}}{z}$$
$$\Rightarrow x = \frac{z \cdot x_{\mathrm{R}}}{f} - \frac{b}{2}$$

follows by differentiation that

$$\frac{\mathrm{d}x}{\mathrm{d}x_{\mathrm{R}}} = \frac{z}{f} \ .$$

Using a pixel size of $p = \frac{D}{1024}$ = 12.4 µm as the accuracy of the camera Δx_{R} and the optical focal length f from Eq. (SI.25), the total accuracy results as

$$\Delta x = \frac{z}{f \cdot \Delta x_{\mathrm{R}}} = 2.45 \text{ mm} \ . \tag{SI.28}$$

In reality, the accuracy is lesser due to

- a residual camera error,
- a non-symmetrically positioned volume,
- a depth variation of the volume, and
- inaccuracies due to tracking markers being finite.

The accuracy can be increased by

- a reduced size of the volume,
- a larger chip and simultaneously larger focal length, and by
- the detection of markers at subpixel level by means of image processing.

4. An ultrasound system with 50 kHz has a wavelength of 6.9 mm in air. It can thus easily compete in terms of accuracy with an optical system.